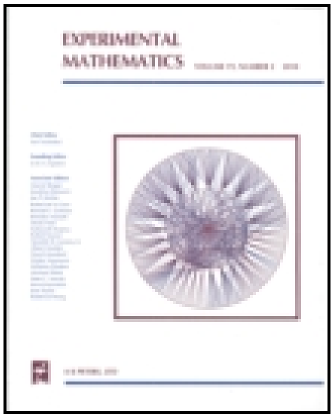

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Computing Maximal Copies of Polyhedra Contained in a Polyhedron

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Computing Maximal Copies of Polyhedra Contained in a Polyhedron

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CONTENTS

Kepler (1619) and Croft (1980) considered the problem of finding the largest homothetic copies of one regular polyhedron contained in another regular polyhedron. For arbitrary pairs of polyhedra, we propose to model this as a quadratically constrained optimization problem. These problems can then be solved numerically; in case the optimal solutions are algebraic, exact optima can be recovered by solving systems of equations to very high precision and then using integer relation algorithms. Croft solved the special cases concerning maximal inclusions of Platonic solids for 14 out of 20 pairs. For the six remaining cases, we give numerical solutions and conjecture exact algebraic solutions.

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1. INTRODUCTION

Given two polyhedra P and Q, we can ask for a polyhedron P' of largest volume such that P' is similar to P and contained in Q. By "similar," we understand that P' can be transformed into P by a dilation and rigid motions. Instead of "largest volume," we might as well ask for a polyhedron that maximizes the dilation factor between P and P'. An equivalent question asks for the smallest polyhedron Q' that is similar to Q and contains P.

The earliest investigation of this topic might be found already in [Kepler 19, libri V, caput I, p. 181]. One finds descriptions of the largest regular tetrahedron included in a cube and of the largest cube included in a regular dodecahedron, although no claim on maximality is made.

A substantial contribution was made in [Croft 80], where the case that P and Q are three-dimensional is considered. Croft notes that apart from exceptional cases, local maxima must be immobile and must satisfy seven linear constraints; see [Croft 80, Theorem, p. 279]. Using this information, he calculates all local maxima and obtains global maximal configurations; see [Croft 80, pp. 283–295]. Letting P and Q range over the Platonic solids, Croft gives a complete answer for 14 out of the 20 nontrivial cases. This is the problem described as [Croft et al. 91, Problem B3, p. 52]; see below for a solution to the remaining six cases.

Containment problems for (simple) polygons are discussed, for example, in [Chazelle 83] and [Agarwal et al. 98], and some algorithms are given. Taking P to be a regular n-gon and Q to be a regular m-gon, the size of the largest copy of P inside Q is known if and only if n and m share a common prime factor. If they are coprime, only conjectural results are known; see [Dilworth and Mane 10].

More general containment problems are studied in [Gritzmann and Klee 94], where the authors also allow groups other than the group of similarities to act on the polyhedra. The problem in which the group acting is the group of similarities appears in [Gritzmann and Klee 94, p. 143], but the authors do not discuss a computational approach.

The related problem of finding largest, not necessarily regular, j-simplices in k-cubes is related to Hardamard matrices and is discussed in [Hudelson et al. 96]. In some cases, the maximizer is indeed a regular simplex; see [Maehara et al. 09] for details.

A short summary of the results of this paper by the author has been posted on MathOverflow [Firsching 14].

In Section 2, we present a method for finding solutions to this problem in general. In the last section, we apply this method to some special cases and thereby solve the above-mentioned Problem B3 numerically and offer conjectural exact algebraic solutions.

2. METHODS

2.1. Setting Up the Optimization Problem

Let P and Q be polyhedra, let p be the dimension of P, and let q be the dimension of Q. We assume $q \geq p$; otherwise, it is not quite clear what it means for P to be included in Q. Let $H_1, \ldots, H_m$ be the defining half-spaces for Q such that

$$Q = \bigcap_{k=1}^{m} H_k,$$

and let $w_1, \ldots, w_n$ denote the vertices of P. We formulate the problem of finding the largest polyhedron P' such that P' is contained in Q and similar to P as a quadratic maximization problem.

Problem 2.1.

Input data: half-spaces $H_1, \ldots, H_m$ of Q, vertices $w_1, \ldots, w_n$ of P.

Variables: s and v_{ij} for $1 \leq i \leq n$, $1 \leq j \leq q$.

Objective function: maximize s.

Linear constraints: $(v_{i1}, \ldots, v_{iq}) \in H_k$ for $1 \leq i \leq n$, $1 \leq k \leq m$.

Quadratic constraints: $\sum_{l=1}^{q}(v_{il} - v_{kl})^2 = s\|w_i - w_j\|_2^2$ for $1 \leq i < j \leq n$.

In this formulation, the variable s can be thought of as the square of the dilation factor between P and P'. The other variables are the coordinates of the vertices of P'. The linear constraints consist of nm weak inequalities. They ensure that we have $P' \subset Q$. The quadratic constraints assert that the distances between vertices of P' agree with those of P up to a dilation factor $\sqrt{s}$, which is the same for all pairs of vertices. Hence the quadratic equalities ensure that P' is similar to P.

A global optimum of the optimization problem gives us a largest polyhedron P', as desired. It might happen that there are combinatorially different optimal solutions to our problem. The goal in Section 3 is to identify one of the optimal solutions. From that, we can deduce the optimal dilation factor and hence determine the size of the largest polyhedron P' similar to P and contained in Q. We do not explain in what combinatorially different ways P' can be contained in Q, but rather describe one possible inclusion.

2.1.1. Improved Formulation

The above formulation for Problem 2.1 is particularly simple and straightforward. However, an equivalent formulation using fewer variables and fewer quadratic constraints can be obtained as follows.

Choose an affine basis from the set of vertices of P. For the optimization problem, we can then take only those variables v_{ij} such that the w_i belong to that affine basis and substitute all occurrences of other variables by linear combinations of the former. These linear combinations can be obtained from the vertices of P, using the fact that we have chosen an affine basis. Using this substitution, we have $(p+1)q+1$ variables in total, and this number depends only on the dimensions of P and Q and not on the number of vertices of P.

In order to obtain fewer quadratic constraints, we also focus on the chosen affine basis: It is enough to ensure that all the distances between all pairs of two vectors in the affine basis are scaled by the same factor $\sqrt{s}$. Because there are $q+1$ vectors in the affine basis, we obtain

$$\binom{q+1}{2} = \frac{1}{2}(q+1)(q+2)$$

quadratic equations. Counting the number of linear equations, we see that there are nm of them, independent of the dimension of Q.

An axis-aligned bounding box for Q gives bounds on the variables v_{ij}. We can trivially include a copy of P, whose circumsphere coincides with the in-sphere of Q, so a lower bound for s would be the square of the Keplerian ratio

$$s \geq \left(\frac{\text{inradius of } Q}{\text{circumradius of } P}\right)^2$$

In a similar way, we could give an upper bound for s, but in view of the objective function, this does not seem necessary.

The equations used in setting up Problem 2.1 depend on the position of Q. If many of the defining hyperplanes for Q are parallel to many coordinate axes, then fewer variables are used in the linear equations. Also, the choice of an affine basis of P might influence the number of variables used in the equations.

In solving Problem 2.1 with a numeric solver, the precision for the input of the polyhedron should be higher than the desired precision.

If P and Q possess symmetry, one can use that symmetry to obtain additional constraints. For example, if P and Q are centrally symmetric, then it suffices to search for a maximal P' among the copies of P that are concentric with Q. See [Croft 80, Observation, p. 288] for a simple proof.

If P and Q are regular polyhedra, one can say without loss of generality that one vertex of P' must lie in one face of Q.

2.1.2. Solving the Optimization Problem Numerically

In order to solve Problem 2.1 numerically, we can use SCIP, which is a solver for mixed integer nonlinear programming. This solver uses branch and bound techniques in order to find a global optimum within a certain precision; see [Achterberg 09] and [Achterberg et al. 08] for details. We do not use SCIP's capability to handle integer variables, because all of our variables are continuous.

2.2. From Numerical Solutions to Exact Solutions

2.2.1. Setting Up the Quadratic System

We obtain approximate results for the global optimum in Problem 2.1 within a certain precision; let us call the resulting polyhedron $\widetilde{P}$. The goal is to derive exact values for the coordinates of a polyhedron P' that is indistinguishable from $\widetilde{P}$ within the precision.

We can identify the vertices of $\widetilde{P}$ that lie in a face of Q. If $\widetilde{P}$ has been calculated with sufficiently high precision (see assumptions in Section 2.3), then $\widetilde{P}$ will satisfy the same vertex–face incidences as an optimal solution P'. In fact, P' is given by the real solution of a system of quadratic equations that is derived from these incidences. An approximate real solution of this system is given by $\widetilde{P}$.

2.2.2. Solving the Quadratic System

A numerical solution to this quadratic system with arbitrary precision can be obtained using Newton's method, and a solution P' to Problem 2.1 gives a good starting point. If all the defining hyperplanes of Q are defined in terms of algebraic numbers, then solutions of the quadratic system must be algebraic. If the system obtained in this way is too complicated to be solved by hand or automatically by a computer algebra system, we can attempt to find solutions using the following three-step approach. We already have an approximate real solution given by $\widetilde{P}$.

1. Numerically approximate the solution to high precision, for example using the multidimensional Newton's method.
2. For each variable, guess the algebraic number close to the approximation using integer-relation algorithms such as LLL [Lenstra et al. 82].
3. Verify the solution by exact calculation in the field of real algebraic numbers.

We can expect to find solutions if they consist of algebraic numbers with minimal polynomials of low degree and small coefficients. See Section 3 for two successful applications of this method. This method can in principle be applied to any given system of equations with algebraic solutions for which we can obtain high-precision numerical approximate solutions.

2.3. Limitations of the Method

The solver SCIP, which can be used for solving Problem 2.1, finds a global optimum, but the calculations are done only with a certain prescribed precision. In general, it might be the case that there exists a maximizer P' that attains the maximal dilation factor $\sqrt{s}$ and a second locally maximal feasible solution P'' with dilation factor $\sqrt{s-\varepsilon}$ for a small $\varepsilon > 0$. Indeed, it is possible to construct examples of P and Q for which that is the case for arbitrarily small ε. Take, for example, both P and Q to be the same rectangle with almost equal side lengths.

Hence in order ensure that we have indeed found an optimal solution to Problem 2.1, we make the following assumptions.

Assumption 2.2. *The solution $\widetilde{P}$ to Problem 2.1 has sufficient precision that there is only one local maximum P' near $\widetilde{P}$.*

Assumption 2.3. *Problem 2.1 has been solved with sufficient precision that the dilation factor $\sqrt{s}$ of the local maximum P' near $\widetilde{P}$ is the global maximum.*

Assumption 2.4. *Problem 2.1 has been solved with sufficient precision that $\widetilde{P}$ and the local maximum P' near $\widetilde{P}$ satisfy the same vertex–face incidences with Q.*

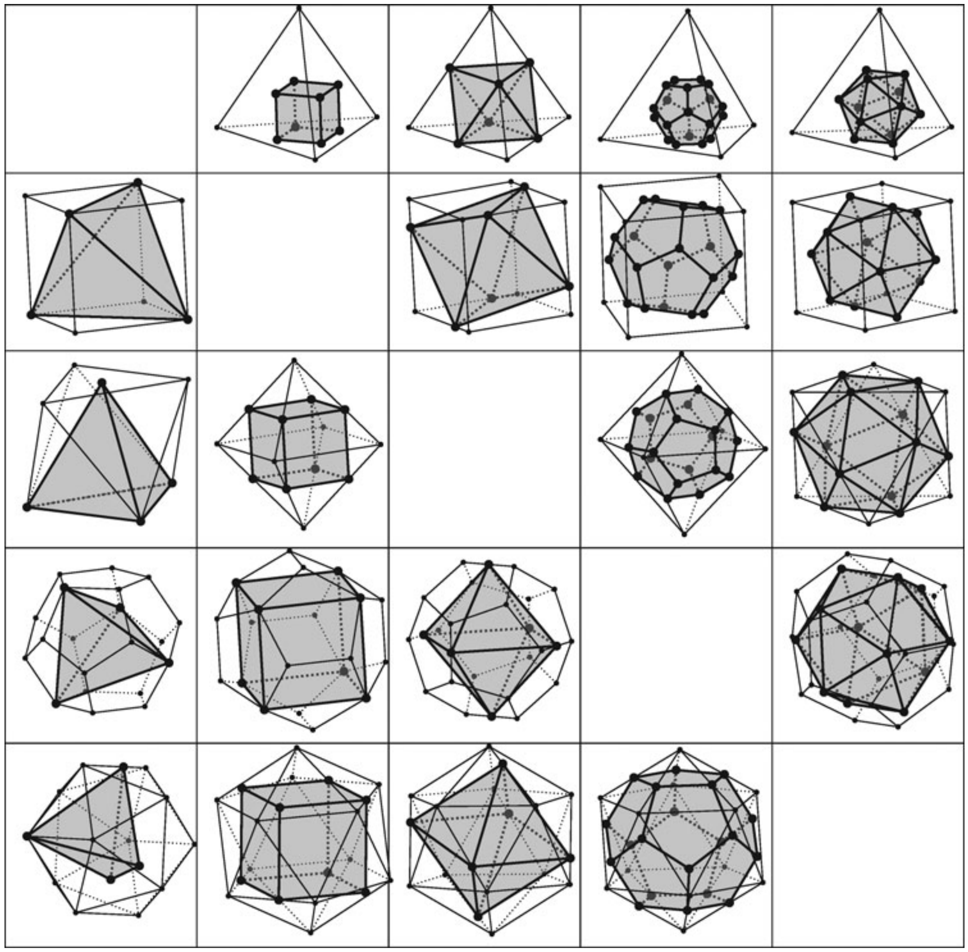

FIGURE 1. Maximal Platonic solids included in a Platonic solid.

The precision necessary for the solution to satisfy these properties depends on P and Q, and because there exist examples for which the global maximum and the second-largest local maximum are arbitrarily close, it is in general impossible to prescribe the precision necessary for Assumptions 2.2–2.4 to hold.

Assumptions 2.2–2.4 also deal with possible numerical mistakes or bugs of a solver for Problem 2.1.

If Assumptions 2.2 and 2.3 hold and we can, because of Assumption 2.4, identify an exact algebraic solution near P', then this will be a maximizer of the problem. In any case, even if the assumptions do not hold, we obtain a lower bound if we can solve the system derived from the approximate solution P'.

In the calculations in Section 3, we do not attempt to prove that Assumptions 2.2–2.4 hold, but we state the precision that was used to solve the problems. In this sense, our calculations below do not prove optimality but provide putatively optimal results.

3. RESULTS

3.1. Inclusions of Platonic Solids

When each of P and Q is taken to be one of the five Platonic solids, that is, one of the regular three-dimensional polyhedra, we can consider 20 nontrivial inclusions. Croft found optimal pairs in 14 out of these 20 cases and proved optimality in [Croft 80]. In the following, we assume that the regular three-dimensional polyhedron Q has side length 1. We abbreviate tetrahedron, cube, octahedron, dodecahedron, and icosahedron by T, C, O, D, and I and denote the golden ratio by ϕ.

With the methods described above, we are able to confirm all the known cases and answer all six unknown cases. The solver used was SCIP version 3.1.0 with a precision set to 10^{-10}. With the improved formulation described above, the calculations for all 20 inclusions took a few hours on a single core of a Xeon CPU running at 3 GHz, using fewer than 8 GB of RAM. Some cases were solved in less than a second.

$Q\backslash P$	T	C	O	D	I
T		0.29590654	0.50000000	$\star$ 0.16263158	0.27009076
C	1.4142136		1.0606602	*0.39428348*	0.61803399
O	1.0000000	0.58578644		$\star$ 0.31340182	0.54018151
D	*2.2882456*	1.6180340	*1.8512296*		$\star$ 1.3090170
I	$\star$ 1.3474429	$\star$ 0.93874890	1.1810180	$\star$ 0.58017873	

TABLE 1. Numerical values.

Tables 1 and 2 give decimal approximations and symbolic values of the side length of a largest copy of P inside Q, where P and Q range over the Platonic solids. For completeness, we restate the results of Croft, who gives a similar but incomplete table [Croft 80, p. 295]. We correct three typos in his table; the correct values are given in italics. New results are marked with a star.

For the six previously unknown cases, we provide below a description of an optimal position.

3.1.1. Dodecahedron in Icosahedron

For D in I, we are in a concentric situation. The five vertices of one face of D lie on the five edges of I incident to a common vertex, one on each. The five vertices of the face opposite that face of D also lie on five edges of I incident to a common vertex, namely the vertex of I antipodal to the one mentioned before. The other ten vertices of D lie in the interiors of faces of I. The side length is

$$\frac{15-\sqrt{5}}{22} \approx 0.58017873.$$

3.1.2. Icosahedron in Dodecahedron

For I in D, we are also in a concentric situation; each of the 12 vertices of I lies in the interior of one of the 12 faces of D, and in each face of D, there is one vertex of I. Let position D in the usual fashion such that six of its edges are parallel to the three coordinate axes.

To each of the 12 vertices on these edges of D, we associate the unique face that contains one but not the other vertex of the edge in its boundary. This gives us pairs v, f of vertices and

$Q\backslash P$	T	C	O	D	I
T		$\left(1+\frac{2}{3}\sqrt{3}+\frac{1}{2}\sqrt{6}\right)^{-1}$	$\frac{1}{2}$	$\star\, d$	$\frac{1}{\phi^2\sqrt{2}}$
C	$\sqrt{2}$		$\frac{3}{4}\sqrt{2}$	$\frac{1}{\sqrt{2}\phi^3}\left(1-\frac{1}{2}\sqrt{10}+\frac{1}{2}\sqrt{2}+\sqrt{5}\right)$	$\frac{1}{\phi}$
O	1	$2-\sqrt{2}$		$\star\, \frac{\left(25\sqrt{2}\right)-\left(9\sqrt{10}\right)}{22}$	$\frac{\sqrt{2}}{\phi^2}$
D	$\phi\sqrt{2}$	ϕ	$\frac{\phi^2}{\sqrt{2}}$		$\star\, \frac{1}{2\phi}+1$
I	$\star\, t$	$\star\, \frac{5+7\sqrt{5}}{22}$	$\frac{1}{2}\left(1-\frac{1}{2}\sqrt{10}+\frac{1}{2}\sqrt{2}+\sqrt{5}\right)$	$\star\, \frac{15-\sqrt{5}}{22}$	

TABLE 2. Exact values.
$\phi =$ golden ratio
$t =$ zero near 1.3 of $5041x^{32} - 1318386x^{30} + 60348584x^{28} - 924552262x^{26} + 5246771058x^{24} - 15736320636x^{22} + 29448527368x^{20} - 37805732980x^{18} + 35173457839x^{16} - 24298372458x^{14} + 12495147544x^{12} - 4717349124x^{10} + 1256858478x^{8} - 217962112x^{6} + 21904868x^{4} - 1536272x^{2} + 160801$
$d =$ zero near 0.16 of $4096x^{16} - 3701760x^{14} + 809622720x^{12} - 17054118000x^{10} + 79233311025x^{8} - 94166084250x^{6} + 31024053000x^{4} - 3236760000x^{2} + 65610000$.

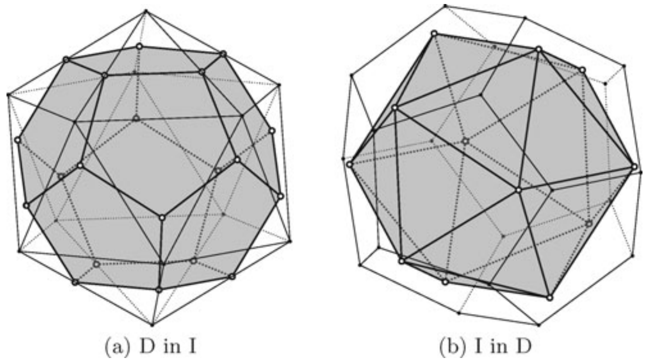

FIGURE 2. Self-reciprocal cases: D in I (left), I in D (right).

faces of D. For each pair v, f, a vertex of I lies on the bisector of f, which goes through v, and its position on the bisector is the point where the bisector is divided into two parts such that the larger part has $\phi/2$ the length of the whole bisector. The position of the vertex of I is closer to v, and the absolute distance to v is

$$\left(1-\frac{\phi}{2}\right)\cdot\frac{1}{2}\sqrt[4]{5}\phi^{3/2}=\frac{\sqrt[4]{5}}{4\sqrt{\phi}}.$$

Recall that we assume that D has side length 1, which results in a bisector of length

$$\frac{1}{2}\sqrt[4]{5}\phi^{3/2}.$$

The edge length of I obtained in this way is equal to

$$\frac{1}{2\phi}+1\approx 1.3090170.$$

3.1.3. Cube in Icosahedron

This is also a concentric situation. For C in I, two vertices of one edge of C lie in the interior of two adjacent edges in I that are not contained in the same face. And the vertices of the antipodal edge of this edge in C lie in the interior of the corresponding antipodal edges in I. The other four edges of C lie in the interiors of faces of I. The side length is

$$\frac{5+7\sqrt{5}}{22}\approx 0.93874890.$$

3.1.4. Dodecahedron in Octahedron

Again, this is a concentric situation. Put two opposite edges of D in a hyperplane spanned by four vertices of O. Four faces of O each contain an edge of D, and the other four faces of O each contain only one vertex of D. The incidences can be seen in the right-hand picture of Figure 3; vertices of D that lie in the interior of a face of O are indicated by hollow circles. See the considerations about reciprocity below. For D in O, the maximum is

$$\frac{\left(25\sqrt{2}\right)-\left(9\sqrt{10}\right)}{22}\approx 0.31340182.$$

Reciprocity of C in I and D in O

If $P\subset Q$ are concentric and P is maximal in Q, we can take polar reciprocals and obtain $Q^\circ\subset P^\circ$ such that Q° is maximal in P°. Because $C^\circ=O$ and $I^\circ=D$, we can check that the two previous cases are reciprocal:

$$\frac{\left(25\sqrt{2}\right)-\left(9\sqrt{10}\right)}{22}\left(\frac{\phi^3}{\sqrt{2}}\right)=\frac{5+7\sqrt{5}}{22}.$$

Concentric C and D that are reciprocals with respect to the unit sphere have the product of their edge lengths constant, namely $2\sqrt{2}$. Similarly, for concentric reciprocal I and D, this product equals $4/\phi^3$. The factor $\phi^3/\sqrt{2}$ is the quotient of these two numbers.

3.1.5. Tetrahedron in Icosahedron

The incidences of T in I are best seen in the left-hand picture of Figure 4: one vertex of T coincides with one vertex v of I, another vertex of T lies on an edge of I, which is neither incident to the vertex v nor its antipode, and the two remaining vertices lie in the interior of faces of I.

While in this case, the resulting system can be somewhat automatically solved by the computer algebra system Mathematica 9 (while version 8 was not able to perform the calculation), we use the methods described in Section 2.2.2. We choose two variables each for the barycentric coordinates for the two vertices in the interior of faces of I and one variable for the barycentric coordinates for the vertex in the interior of an edge of I. Together with a variable t for the side length of T, i.e., the dilation factor, this results in a system of six quadratic

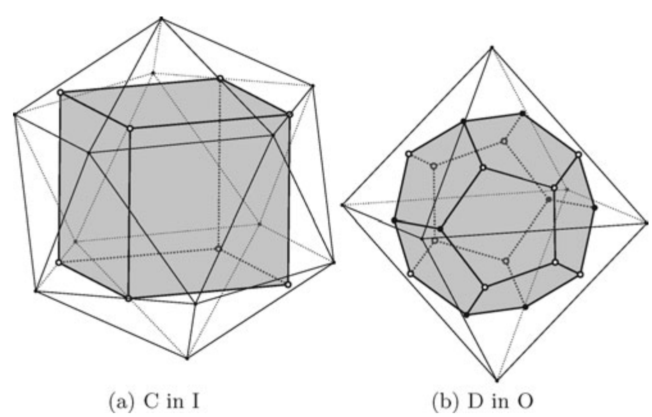

FIGURE 3. Two reciprocal cases: C in I (left), D in O (right).

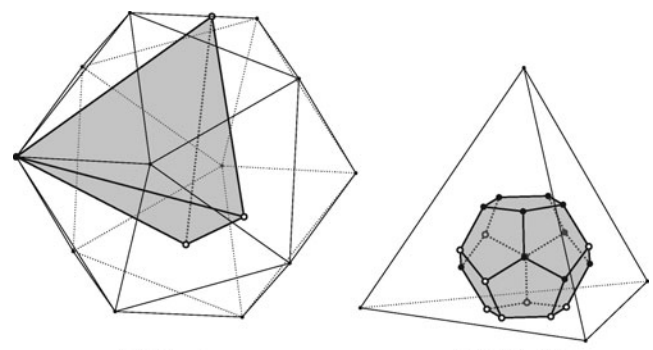

FIGURE 4. Two cases with more involved solutions: T in I (left), D in T (right).

equations in six variables. The six equations confirm that all six edges are of length t.

We use the open-source computer algebra system Sage.[1] For the Newton method, i.e., step 1, we use mpmath,[2] and for the integer relation, i.e., step 2, PARI is used.[3] It is sufficient to obtain 800 decimal digits in step 1 of the method described in Section 2.2.2 in order to obtain the exact values for the variables in step 2. The exact edge length is the zero near 1.3474429 of this polynomial:

$$\begin{aligned}&5041t^{32} - 1318386t^{30} + 60348584t^{28} - 924552262t^{26}\\&+ 5246771058t^{24} - 15736320636t^{22} + 29448527368t^{20}\\&- 37805732980t^{18} + 35173457839t^{16} - 24298372458t^{14}\\&+ 12495147544t^{12} - 4717349124t^{10} + 1256858478t^{8}\\&- 217962112t^{6} + 21904868t^{4} - 1536272t^{2} + 160801.\end{aligned}$$

3.1.6. Dodecahedron in Tetrahedron

The incidences are best seen in the right-hand picture in Figure 4: a complete face of D is contained in one face of T, two vertices of D lie in another face of T, and the two other faces of T contain one vertex of D each. We choose a variable d for the side length of D and four additional variables that describe the position of the vertices of D that lie in a face of T that is not the face that contains a complete face of D. Making sure that the edges between these four vertices have the correct length results again in a system of six quadratic equations in five variables, which can be successfully solved as in the previous case. In this case, 350 decimal digits suffice to find solutions in the field of real algebraic numbers. The exact edge length is the zero near 0.16263158 of this polynomial:

$$\begin{aligned}&4096d^{16} - 3701760d^{14} + 809622720d^{12} - 17054118000d^{10}\\&+ 79233311025d^{8} - 94166084250d^{6} + 31024053000d^{4}\\&- 3236760000d^{2} + 65610000.\end{aligned}$$

4. FURTHER APPLICATIONS

Possibly interesting situations to which the method of this paper could be applied include the following cases.

(a) Take P and Q to be (regular) polygons.

(b) Take P and Q to be regular polyhedra of dimension greater than 3.

(c) Take P to be an n-cube and Q an m-cube with $n < m$.

(d) Take P to be a regular n-simplex and Q an m-cube with $n \leq m$.

(e) Take Q to be any polyhedron and P some projection of Q.

For the first case, i.e., finding the largest regular n-gon in a regular m-gon, the author has checked the conjecture of [Dilworth and Mane 10, Section-9] for coprime m and n up to a precision of 10^{-10} for all pairs m, n with $m, n \leq 120$. It is possible to modify Problem 2.1 to solve similar packing problems.

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REFERENCES

[Achterberg 09] Tobias Achterberg. "SCIP: Solving Constraint Integer Programs." *Mathematical Programming Computation* 1 (2009), 1–41.

[Achterberg et al. 08] Tobias Achterberg, Timo Berthold, Thorsten Koch, and Kati Wolter. "Constraint Integer Programming: A New Approach to Integrate CP and MIP." In *Integration of AI and OR Techniques in Constraint Programming for Combinatorial Optimization Problems*, edited by Laurent Perron and Michael A. Trick, Lecture Notes in Computer Science 5015, pp. 6–20. Springer, 2008.

[1] Available at http://www.sagemath.org.
[2] Available at http://mpmath.org/.
[3] Available at http://pari.math.u-bordeaux.fr/.

[Agarwal et al. 98] Pankaj K. Agarwal, Nina Amenta, and Micha Sharir. "Largest Placement of One Convex Polygon Inside Another." *Discrete and Computational Geometry* 19 (1998), 95–104.

[Chazelle 83] Bernard Chazelle. "The Polygon Containment Problem." In *Advances in Computing Research I*, edited by Franco P. Preparata, pp. 1–33. JAI Press, 1983.

[Croft 80] Hallard T. Croft. "On Maximal Regular Polyhedra Inscribed in a Regular Polyhedron." *Proceedings of the London Mathematical Society* 3 (1980), 279–296.

[Croft et al. 91] Hallard T. Croft, Kenneth J. Falconer, and Richard K. Guy. *Unsolved Problems in Geometry*. Springer, 1991.

[Dilworth and Mane 10] Stephen J. Dilworth and Sateesh R. Mane. "On a Problem of Croft on Optimally Nested Regular Polygons." *Journal of Geometry* 99 (2010), 43–66.

[Firsching 14] Moritz Firsching. "On Maximal Regular Polyhedra Inscribed in a Regular Polyhedron." MathOverflow, 2014 (version: 2014-01-09). Available online (http://mathoverflow.net/q/154007).

[Gritzmann and Klee 94] Peter Gritzmann and Victor Klee. "On the Complexity of Some Basic Problems in Computational Convexity: I. Containment Problems." *Discrete Mathematics* 136 (1994), 129–174.

[Hudelson et al. 96] Matthew Hudelson, Victor Klee, and David Larman. "Largest j-Simplices in d-Cubes: Some Relatives of the Hadamard Maximum Determinant Problem." In *Linear Algebra and Its Applications*, pp. 519–598. Proceedings of the Fourth Conference of the International Linear Algebra Society, 1996.

[Kepler 19] Johannes Kepler. *Harmonices Mundi*. 1619.

[Lenstra et al. 82] Arjen K. Lenstra, Hendrik W. Lenstra, Jr., and László Lovász. "Factoring Polynomials with Rational Coefficients." *Mathematische Annalen* 261 (1982), 515–534.

[Maehara et al. 09] Hiroshi Maehara, Imre Z. Ruzsa, and Norihide Tokushige. "Large Regular Simplices Contained in a Hypercube." *Periodica Mathematica Hungarica* 58 (2009), 121–126.