Diplomarbeit

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Reeller äquivarianter Bordismus für Produkte von $\mathbb{Z}/2$

(Real equivariant bordism for products of $\mathbb{Z}/2$)

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[1] siehe Bemerkung auf der nächsten Seite

[2] see remark on the next page

Bermerkung. Dies ist eine revidierte Version, in welcher zahlreiche Fehler aller Art korrigiert sind. Auf die meisten dieser Fehler wurde ich von Peter Landweber aufmerksam gemacht, welchem ich sehr dankbar für seine nützlichen Anmerkungen und Verbesserungen bin.

Remark. This is a revised version, which corrects numerous errors of many kinds. Most of these errors were brought to my attention by Peter Landweber and I am most grateful for his valuable comments and corrections.

Contents

Einleitung

Zuerst geben wir eine kurze allgemeine Einführung zu den Themen, die in dieser Arbeit behandelt werden. Wir nennen unser Hauptresultat und erklären die Inhalte der einzelnen Kapitel.

Äquivarianter Bordismus ist eine Methode, Mannigfaltigkeiten mit Gruppenwirkung besser zu verstehen. Es gibt zwei verschiedene Zugänge zu Bordismus; einen geometrischen und einen homotopietheoretischen. Für eine fest gewählte Gruppe G wird geometrischer äquivarianter Bordismus durch eine Äquivalenzrelation auf geschlossenen Mannigfaltigkeiten mit G-Wirkung definiert. Homotopietheoretischer äquivarianter Bordismus ist die zum äquivarianten Thomspektrum assoziierte Homologietheorie. Die Pontrjagin[1]–Thom Konstruktion gibt eine Abbildung von geometrischem in den homotopietheoretischen Bordismus. Im nicht-äquivarianten Fall, daß heißt falls G die triviale Gruppe ist, ist diese Abbildung ein Isomorphismus. Falls G nicht-trivial ist, so ist die Pontrjagin–Thom Abbildung kein Isomorphismus. Dadurch wird die Beschreibung von *äquivariantem* Bordismus komplizierter.

Es gibt reelle und komplexe Mannigfaltigkeiten und reelle und komplexe Vektorbündel; das führt zu reellen und komplexen Bordismustheorien. Wir konzentrieren uns auf *reellen* Bordismus und die Gruppen, für die wir uns interessieren, sind Produkte von $\mathbb{Z}/2$, also $G = (\mathbb{Z}/2)^k$ für ein k.

Unsere Beschreibung von äquivariantem Bordismus für $G = \mathbb{Z}/2 \times \cdots \times \mathbb{Z}/2$ führt zu dem Hauptresultat, Theorem 3.5.2, welches hier schon einmal genannt werden soll.

Satz. *Das folgende Diagramm ist ein Pullback-Quadrat, in dem alle Abbildungen injektiv sind.*

$$\begin{tikzcd} \mathfrak{N}_*^G \arrow[r, "\phi_{\mathfrak{N}}"] \arrow[d, "PT"] & MO_*[e_V^{-1}, Y_{d,V}] \arrow[d, "i"] \\ MO_*^G \arrow[r, "\iota\circ\phi_{MO}"] & MO_*[e_V, e_V^{-1}, Y_{d,V}]. \end{tikzcd} \tag{$\circledast$}$$

Links stehen geometrischer und homotopietheoretischer reeller äquivarianter Bordismus und die senkrechte Abbildung ist die oben erwähnte Pontrjagin–Thom Abbildung. Auf der rechten Seite stehen Polynomringe und als senkrechte Abbildung eine Inklusion. Die Polynomringe sind über MO_*, dem reellen nicht-äquivariantem Bordismusring. Dessen Struktur wird in Kapitel 2 erklärt. Theorem 3.5.2 ist die reelle Version eines Satzes von Hanke [Han05, Theorem 1] und wir benutzen die gleichen Mittel wie er in seinem Beweis.

Kapitel 1 ist eine Einführung in alle Begriffe, die nötig sind, um den Rest der Arbeit zu formulieren. Zuerst rufen wir die Grundlagen äquivarianter Topologie in Erinnerung. Dann stellen wir die grundlegenden Ideen äquivarianter Spektra und (Ko)homologietheorien dar. Hier wählen wir Mays Zugang (siehe etwa [MPC96], [LMS86]), der ein vollständiges G-Universum, welches alle Isomorphieklassen reeller G-Darstellungen enthält, für die Indizierung benutzt. Diese Werkzeuge erlauben uns, das reelle äquivariante Thomspektrum und damit homotopietheoretischen äquivarianten Bordismus zu definieren. Nachdem auch geometrischer äquivarianter Bordismus definiert wird, skizzieren wir die äquivariante Pontrjagin–Thom Konstruktion, welche eine Abbildung von geometrischem in den

[1] Umschrift seines russischen Namens Понтрягин.

homotopietheoretischen Bordismus liefert. Die Begriffe „Familien von Untergruppen“ und Homologietheorien assoziiert zu solchen Familien werden vorgestellt und das führt auf zwei exakte Sequenzen: der Conner–Floyd Sequenz im geometrischem Bordismus und der tom Dieck Sequenz im homotopietheoretischem Bordismus. Die Tatsache, daß die Pontrjagin–Thom Abbildung für nicht-triviale Gruppen nicht surjektiv, ist folgt aus Proposition 1.8.2, in der Bedingungen dafür genannt werden, daß gewisse Elemente aus homotopietheoretischem äquivariantem Bordismus, genannt Eulerklassen, nicht-trivial sind.

In diesem Kapitel werden sowohl reeller als auch komplexer Bordismus definiert, aber der reelle Fall wird in größerer Ausführlichkeit behandelt; erstens weil Theorem 3.5.2 eine Aussage über *reellen* Bordismus ist und zweitens weil der komplexe Bordismus bereits besser in der Literatur beschrieben ist.

Im zweiten Kapitel nennen wir einige bekannte Ergebnisse ohne deren Beweise zu wiederholen. Wir fangen mit der nicht-äquivarianten Pontrjagin–Thom Abbildung an und fahren mit reellem äquivariantem Bordismus fort. Die Ergebnisse von Sinha [Sin02] für $G = \mathbb{Z}/2$ werden erwähnt, weil sie einen Spezialfall für die allgemeinere Annahme $G = \mathbb{Z}/2 \times \cdots \times \mathbb{Z}/2$ darstellen. Das Kapitel endet mit Ergebnissen über die Injektivität von der reellen und komplexen Pontrjagin–Thom Abbildung in bestimmten Fällen.

Das Ziel von Kapitel 3 ist es, Hankes Arbeit [Han05] aus dem komplexen in die reelle Umgebung zu übersetzen. Dafür müssen wir die Gruppe, die Hanke benutzt, nämlich den n-Torus $S^1 \times \cdots \times S^1$, durch die Gruppe $\mathbb{Z}/2 \times \cdots \times \mathbb{Z}/2$ ersetzen. (S^1 ist Retrakt von $\mathbb{C}^\times$ genauso wie $\mathbb{Z}/2$ Retrakt von $\mathbb{R}^\times$ ist; das läßt diese Veränderung plausibel erscheinen.) Die erste Schwierigkeit ist, die waagerechten Abbildungen in ($\circledast$) und die Polynomringe auf der rechten Seite zu definieren. Dafür wird eine Einschränkung auf Fixpunktmengen benutzt, in einer Art, die auf tom Dieck [tD70] zurückgeht. Die Injektivität der unteren waagerechten Abbildung in ($\circledast$) ist eine Folgerung aus Lokalisierungsargumenten, welche in Abschnitt 3.2 besprochen werden. Das sind alle Instrumente, die für den Beweis von Theorem 3.5.2 benötigt werden. Die Beweisidee ist, das Quadrat als Teil der Pontrjagin–Thom Abbildung zwischen der exakten Sequenz von Conner–Floyd und der exakten Sequenz von tom Dieck zu identifizieren. Dann wird die Exaktheit dieser Sequenzen zusammen mit Ergebnissen über Injektivität bestimmter Abbildungen in einer Diagrammjagd benutzt. Als Korollar erhalten wir eine Beschreibung des geometrischen reellen äquivarianten Bordismusrings. Das Kapitel endet mit einem Vergleich von unseren Ergebnissen mit denen von Sinha [Sin02] für den Fall $G = \mathbb{Z}/2$.

Kapitel 4 zeigt die Grenzen unseres Satzes. Insbesondere zeigen wir durch zwei Gegenbeispiele, daß Theorem 3.5.2 versagt falls G nicht die Gestalt $\mathbb{Z}/2 \times \cdots \times \mathbb{Z}/2$ hat. Im letzten Kapitel sammeln wir einige unbeantwortete Fragen, die sich aus den vorherigen Ergebnissen ergeben.

Für die gute Betreuung durch Professor Jens Hornbostel möchte ich mich herzlich bedanken; er hat sich immer Zeit für meine Fragen genommen und mir gleichzeitig die Möglichkeit gegeben, eigenständig zu arbeiten. Besonders dankbar bin ich dafür, daß er mich ermutigt hat, ein Jahr in Paris zu verbringen und für seine Zusammenarbeit mit Professor Bob Oliver während dieser Zeit.

Introduction

First we give a short general introduction to the subjects treated in this text. We state our main result and explain the content of the individual chapters.

Equivariant bordism is a tool in the study of manifolds with a group action. There is a geometric and a homotopic approach to bordism. After fixing a compact Lie group G, geometric equivariant bordism is defined in terms of equivalence classes of closed manifolds with G acting on it. Homotopic equivariant bordism is the homology theory associated to the equivariant Thom spectrum. An assignment, called the Pontryagin[1]–Thom construction gives rise to a map from geometric to homotopic equivariant bordism. In the non-equivariant case, i.e. if G is the trivial group, this map is an isomorphism. If G is non-trivial, the Pontryagin–Thom map is not an isomorphism. This is what makes the description of equivariant bordism complicated.

There are real and complex manifolds and there are real and complex vector bundles; this leads to real and complex bordism theories. In this text we focus on *real* bordism and the groups of interest are products of $\mathbb{Z}/2$, i.e. $G = (\mathbb{Z}/2)^k$ for some k.

Our characterization of equivariant bordism for $G = \mathbb{Z}/2 \times \cdots \times \mathbb{Z}/2$ culminates in Theorem 3.5.2 and we state it already here for convenience.

Theorem. *The following diagram is a pull-back with all maps injective:*

$$\begin{tikzcd} \mathfrak{N}_*^G \arrow[r, "\phi_{\mathfrak{N}}"] \arrow[d, "PT"] & MO_*[e_V^{-1}, Y_{d,V}] \arrow[d, "i"] \\ MO_*^G \arrow[r, "\iota\circ\phi_{MO}"] & MO_*[e_V, e_V^{-1}, Y_{d,V}]. \end{tikzcd} \tag{$\circledast$}$$

On the left hand side we have geometric and homotopic real equivariant bordism and the Pontryagin–Thom map between them. On the right hand side there are certain polynomial rings and the map between them is an inclusion. The polynomial rings are over MO_*, the real non-equivariant bordism ring. This ring is well understood as explained in Chapter 2. Our Theorem 3.5.2 is the real version of a theorem by Hanke [Han05, Theorem 1] and we use the same techniques he uses in his proof.

Chapter 1 is an introduction to all the concepts needed to formulate the rest of the text. First we recall the basics of equivariant topology. We then present the fundamental concepts of equivariant spectra and (co)homology theories. Here we choose May's approach (as in [MPC96] and [LMS86]), using a complete G-universe containing all isomorphism classes of G-representations for indexing. These tools allow us to define the real equivariant Thom spectrum and homotopic real equivariant bordism. After defining geometric real equivariant bordism we sketch the equivariant Pontryagin–Thom construction, which gives a map from geometric to homotopic real equivariant bordism. The notion of families of subgroups and homology theories associated to those families of subgroups is defined and this leads to Proposition 1.8.2, which gives conditions under what circumstances certain elements in homotopic equivariant bordism, called Euler classes, are trivial.

[1] Transliteration of his Russian name Понтрягин.

Both real and complex bordism theories are defined in this chapter, but the real case is always discussed in more detail. This is because we will need *real* equivariant bordism in our theorem and also because the complex theory is already better documented in the literature.

In the second chapter we state some known results without giving proofs. We begin with the non-equivariant Pontryagin–Thom map and continue with real equivariant bordism. The results of Sinha [Sin02] for $G = \mathbb{Z}/2$ are mentioned, since they constitute a special case of our more general assumption $G = \mathbb{Z}/2 \times \cdots \times \mathbb{Z}/2$. Results about the injectivity of the real and complex equivariant Pontryagin–Thom map for certain groups conclude the chapter.

The aim of Chapter 3 is to translate Hanke's paper [Han05] from the complex into the real setting. This requires changing the group Hanke uses, namely the n-torus $S^1 \times \cdots \times S^1$ to the group $\mathbb{Z}/2 \times \cdots \times \mathbb{Z}/2$. (Note that S^1 is a retract of $\mathbb{C}^\times$ and $\mathbb{Z}/2$ is a retract of $\mathbb{R}^\times$, so this change might seem plausible.) The first difficulty is to define the horizontal maps in ($\circledast$) and the polynomial rings on the right hand side. Here restriction to fixed sets is used, in a form that goes back to a paper by tom Dieck [tD70]. Injectivity of the lower horizontal map in ($\circledast$) is a consequence of localization results that are discussed in Section 3.2. Then we consider equivariant bordism with respect to families.

This gives us all the instruments we need to complete the proof of Theorem 3.5.2. The idea of the proof is to identify the square ($\circledast$) as part of the diagram that is the Pontryagin–Thom map between the Conner–Floyd exact sequence and the tom Dieck exact sequence. Then the exactness of these sequences and injectivity results are used in a diagram chase. As a corollary we get a description of the geometric real equivariant bordism ring. The chapter closes with a comparison of our result with Sinha's [Sin02] for $G = \mathbb{Z}/2$.

Chapter 4 shows the limitations of our main result. In particular we show by two counterexamples that it fails if G is not of the form $\mathbb{Z}/2 \times \cdots \times \mathbb{Z}/2$.

In the last chapter we point out some unanswered questions that can be asked in view of the prior results.

I would like to thank my advisor Professor Jens Hornbostel. He always supported me by listening to my questions, while giving me the freedom to work on my own. I am especially grateful for his encouragement to spend a year in Paris and his cooperation with Professor Bob Oliver during that time.

1 Basics and Notation

This chapter is a swift introduction to equivariant geometric and homotopic bordism. After introducing the notation of basic concepts of equivariant topology we go on to explain how equivariant spectra and equivariant (co)homology theories are interrelated. Finally the basic notions of equivariant bordism are exhibited. We present both real and complex bordism, but since the real case is more important to us, we sometimes only fix notation and give references for the complex case.

1.1 Equivariant topology

Our introduction follows and uses the notation of Chapter I of [MPC96]. By $\mathscr{U}$ we denote the category of (unbased) compactly generated spaces (in the sense of May [May99, Chapter 5]) and continuous maps. By $\mathscr{T}$ we denote the category of based (or pointed) compactly generated spaces and continuous maps that preserve the base point, i.e. pointed maps. When we talk about "(based) spaces" and "maps" we usually mean objects and morphisms in one of these categories. By $\mathcal{A}b$ we denote the category of Abelian groups. Let G be a topological group. Notice that G can be viewed as a based space by taking the identity to be the base point.

A **G-space** is a space X together with a map $G \times X \to X$ such that

$$ex = x$$

for the identity e of G and

$$g(hx) = (gh)x$$

for all $g, h \in G, x \in X$. For based spaces the map $G \times X \to X$ is assumed to fix the base point $*$ of X, i.e. $g* = *$ for all $g \in G$. The space consisting of one point with the only possible G-action will sometimes be denoted by pt. A map $\phi : X \to Y$ between two G-spaces is **equivariant** or a **G-map** if $\phi(gx) = g\phi(x)$ for all $g \in G, x \in X$. This results in two categories. By $G\mathscr{U}$ we denote the category of (unbased) G-spaces and equivariant maps. By $G\mathscr{T}$ we denote the category of based G-spaces, such that the G-action fixes the base point and pointed equivariant maps. If it is clear from context we will sometimes just say "map" instead of "G-map". Given an unbased G-space X, we denote its disjoint union with a base point with trivial G-action by X_+. This gives a based G-space.

Constructions in $G\mathscr{U}$ and $G\mathscr{T}$ are similar to those in the non-equivariant categories, we will shortly mention a few. For a thorough introduction see tom Dieck's book [tD87]. For a family $\{X_i\}_{i\in I}$ of G-spaces we define a G-action on the product of spaces $\prod_{i\in I} X_i$ by

$$g(x_i \,|\, i \in I) := (gx_i \,|\, i \in I).$$

This is called the diagonal action. We write $\mathrm{Map}(X, Y)$ for the space of maps $X \to Y$, with the compact-open topology (see [May99, Chapter 3]). This can be given the structure of a G-space by conjugation:

$$(g\phi)(x) := g\phi(g^{-1}x)$$

for $\phi \in \mathrm{Map}(X, Y)$. Notice that there is a G-homeomorphism by the usual adjunction

$$\mathrm{Map}(X \times Y, Z) \to \mathrm{Map}(X, \mathrm{Map}(Y, Z)).$$

Given two based G-spaces X and Y we can identify their base points to obtain the wedge $X \vee Y$, again a based G-space. We define the smash product to be

$$X \wedge Y := X \times Y / X \vee Y.$$

We write $\mathrm{F}(X, Y)$ for the based space of based maps $X \to Y$. This can again be given the structure of a based G-space by conjugation. Similarly we have a based G-homeomorphism

$$\mathrm{F}(X \wedge Y, Z) \to \mathrm{F}(X, \mathrm{F}(Y, Z)).$$

For a G-space X the isotropy group of a point $x \in X$ is the subgroup

$$G_x = \{g \in G \mid gx = x\}$$

of G. Two subgroups H and K of a group are called conjugate if there is a $g \in G$ such that $H = gKg^{-1}$. In that case we write $H \sim K$.

We assume all subgroups of G to be closed. For a subgroup $H \subset G$ we have the fixed point set

$$X^H := \{x \in X \mid hx = x \text{ for all } h \in H\}.$$

A G-map $f : X \to Y$ induces maps $f^H : X^H \to Y^H$ for every subgroup $H \subset G$. A homotopy between unbased G-spaces is a G-map $h : X \times [0,1] \to Y$, where $[0,1]$ is given the trivial G-action. This notion of homotopy gives us the homotopy category $hG\mathscr{U}$. A weak equivalence is a G-map $f : X \to Y$ such that $f^H : X^H \to Y^H$ is a weak equivalence in the category of spaces for all subgroups $H \subset G$, i.e. it induces an isomorphism on all homotopy groups. This definition can be extended to a model structure on $G\mathscr{U}$. Cofibrations are defined by the homotopy extension property, fibrations by the covering homotopy property. All maps are taken to be equivariant. By formally inverting the weak equivalences we obtain the category $\overline{h}G\mathscr{U}$. For more details see [MPC96, Chapter IV], for the model structures see [MM02, Chapter III.1]. For based G-spaces a homotopy is a map $h : X \wedge [0,1]_+ \to X$, where $[0,1]_+$ is given the trivial G-action. As in the unbased case one defines the categories $hG\mathscr{T}$ and $\overline{h}G\mathscr{T}$.

Definition 1.1.1. For two objects X and Y in $\overline{h}G\mathscr{T}$ we denote the set of morphisms from X to Y in $\overline{h}G\mathscr{T}$ by $[X, Y]_G$.

There is a concept of equivariant CW-complexes, that is very similar to ordinary CW-complexes. A G-complex X is a CW-complex which is also a G-space such that X^H is a subcomplex for any subgroup $H \subset G$. Details can be found in [MPC96, Chapter I.3].

1.2 Equivariant spectra and (co)homology theories

From now on, let G be a compact Lie group.

We follow [MPC96, Chapters IX and XIII] and [LMS86, Chapters I and II].

Equivariant spectra

Definition 1.2.1. A real (resp. complex) G-**representation** $V = (V, \rho)$, i.e. a homomorphism of Lie groups $\rho : G \to O(V)$ (resp. $\rho : G \to U(V)$).

We sometimes just write G-representation or representation instead of "real or complex G-representation" relying on context for what is meant.

Definition 1.2.2. For a G-representation V, the one-point compactification of V is denoted by S^V. The **unit disc** of the representation is

$$D(V) := \{v \in V \mid |v| \leq 1\}.$$

The **unit sphere** of the representation is

$$S(V) := \{v \in V \mid |v| = 1\}.$$

For a G-representation V, the **projective space** $P(V)$ is the space of one-dimensional subspaces. Compare tom Dieck [tD87, Chapter V, 2.8].

Remark 1.2.3. For a G-representation V, $D(V)$, $S(V)$ and $P(V)$ are G-spaces and S^V is a based G-space. We can give S^V the structure of a G-complex. If V is a finite dimensional representation, S^V can be given the structure of a finite G-complex.

Definition 1.2.4. For a G-representation V of G and a based G-space X we define the suspension

$$\Sigma^V X := X \wedge S^V$$

and the loop space

$$\Omega^V X := F(S^V, X).$$

Remark 1.2.5. Σ^V and Ω^V can be extended to functors $G\mathscr{T} \to G\mathscr{T}$ and Σ^V is left adjoint to Ω^V.

Definition 1.2.6. A G**-universe** $\mathcal{U}$ is a countable direct sum of G-representations such that

1. $\mathcal{U}$ contains a trivial representation and
2. $\mathcal{U}$ contains each of its sub-representations infinitely often.

A G-universe is said to be **complete** if it contains every irreducible representation of G. Depending on whether we take all real or all complex G-representations we obtain a real or a complex G-universe.

Remark 1.2.7. A real G-universe $\mathcal{U}$ can be written as $\bigoplus_{i\in I} V_i^\infty$, where $\{V_i\}_{i\in I}$ is a set of irreducible representations and V_1 is $\mathbb{R}$, the trivial representation. If G is finite, a complete G-universe is given by V^∞, where V is the regular representation of G.

Definition 1.2.8. For a natural number n, we denote the representation $\mathbb{R}^n$ with trivial G-action by $\underline{n}$, sometimes viewed as a subspace of a given G-universe. If there is no possibility of confusion we denote that representation also with n.

Definition 1.2.9. An **indexing space** in $\mathcal{U}$ is a finite dimensional G-subspace of a universe $\mathcal{U}$.

Definition 1.2.10. A G**-prespectrum** $E = (EV, \sigma_{V,W})$ indexed on a G-universe $\mathcal{U}$ is a family of (based) G-spaces EV, one for every indexing space V in $\mathcal{U}$, together with structure maps

$$\sigma_{V,W} : \Sigma^{W-V} EV \to EW$$

for $V \subset W$, such that

$$\begin{tikzcd} \Sigma^{Z-W}\Sigma^{W-V} EV \arrow[r, "\Sigma^{Z-W}\sigma_{V,W}"] \arrow[d, "\cong"] & \Sigma^{Z-W} EV \arrow[d, "\sigma_{W,Z}"] \\ \Sigma^{Z-V} EV \arrow[r, "\sigma_{V,Z}"] & EZ \end{tikzcd}$$

commutes and $\sigma_{V,V} = \mathrm{id}$. Here $W - V$ denotes the orthogonal complement of V in W.

Definition 1.2.11. An Ω-G**-spectrum** is a G-prespectrum such that the adjoints of the structure maps,

$$\tilde{\sigma}_{V,W} : EV \to \Omega^{W-V} EW$$

are weak equivalences. A G**-spectrum** is a G-prespectrum such that the adjoints of the structure maps,

$$\tilde{\sigma}_{V,W} : EV \to \Omega^{W-V} EW$$

are homeomorphisms.

Definition 1.2.12. A map $f : (DV, \rho_{V,W}) \to (EV, \sigma_{V,W})$ of G-prespectra is a family $\{f_V : D_V \to E_V\}$ such that

$$\begin{array}{ccc} \Sigma^{W-V} DV & \xrightarrow{\rho_{V,W}} & DW \\ \downarrow{\scriptstyle \Sigma^{W-V} f_V} & & \downarrow{\scriptstyle f_W} \\ \Sigma^{W-V} EV & \xrightarrow{\sigma_{V,W}} & EW \end{array}$$

commutes for all indexing spaces $V \subset W$. The resulting category of G-prespectra indexed on a G-universe $\mathcal{U}$ is denoted by $G\mathscr{P}\mathcal{U}$ or $G\mathscr{P}$. Maps of Ω-G-spectra and G-spectra are maps of their underlying G-prespectra and we obtain the categories $\Omega\text{-}G\mathscr{S}\mathcal{U} = \Omega\text{-}G\mathscr{S}$ of Ω-G-spectra and $G\mathscr{S}\mathcal{U} = G\mathscr{S}$ of G-spectra.

For $G = \{e\}$, the trivial group, we can choose $\mathbb{R}^\infty$ as complete G-universe and obtain $\{e\}\mathscr{S}\mathbb{R}^\infty$ the category of spectra, which will be denoted by $\mathscr{S}$. Notice that every G-spectrum is an Ω-G-spectrum and every Ω-G-spectrum is a G-prespectrum. In fact we have a forgetful functor:

$$l : G\mathscr{S} \to G\mathscr{P}.$$

This functor has a left adjoint spectrification functor

$$L : G\mathscr{P} \to G\mathscr{S}.$$

The adjunction can be constructed in the same way as non-equivariantly and is described in [MPC96, Chapter XII] and also in [LMS86, Appendix, p. 475] and [EKMM07, p. 10].

Definition 1.2.13. Let E be a G-prespectrum and X be a G-space. The **smash product** $E \wedge X$ is the G-prespectrum with spaces

$$(E \wedge X)V := EV \wedge X$$

for an indexing space V in a G-universe $\mathcal{U}$ and structure maps

$$\sigma \wedge 1 : \Sigma^{W-V}(E \wedge X)V = \Sigma^{W-V}(EV \wedge X) = (\Sigma^{W-V} EV) \wedge X \to EW \wedge X$$

for indexing spaces $V \subset W$ in $\mathcal{U}$. For a G-spectrum F we define

$$F \wedge X := L(lF \wedge X).$$

The smash products $X \wedge E$ and $X \wedge F$ are defined analogously.

The definition of smash products allows to define a notion of homotopy and the category $hG\mathscr{S}\mathcal{U}$, whose morphisms are denoted by $[-,-]_G$. A homotopy in $G\mathscr{S}\mathcal{U}$ is a map $E \wedge I_+ \to F$, where I is the unit interval with trivial G-action. This gives the homotopy category $hG\mathscr{S}\mathcal{U}$. Adjoining formal inverses to weak equivalences gives the category $\bar{h}G\mathscr{S}\mathcal{U}$.

For two G-spectra E and F, the set of homotopy classes $[E, F]_G$ is given the structure of an Abelian group in the usual way. (See for example [Swi75, Corollary 8.27, p. 142].)

Definition 1.2.14. Given a linear isometric isomorphism $f : \mathcal{U} \to \mathcal{U}'$ between two G-universes $\mathcal{U}$ and $\mathcal{U}'$, the **change of universe functor**

$$f^* : G\mathscr{S}\mathcal{U}' \to G\mathscr{S}\mathcal{U}$$

is given by

$$(f^* E')V = E'(f(V))$$

and maps

$$\begin{aligned} \sigma_{V,W} : (f^* E')(V) &= \Sigma^{W-V} E'(f(V)) \\ &\cong \Sigma^{f(W)-f(V)} E'(f(V)) \xrightarrow{\sigma_{f(V),f(W)}} E'(f(W)) = (f^* E')W. \end{aligned}$$

It is possible to define a left adjoint f_* to f^* even when f fails to be an isomorphism, (compare [LMS86, Definition 1.1 and Proposition 1.2, p. 58]).

Given a G-universe $\mathcal{U}$ we can consider the set

$$D := \{V \oplus W \subset \mathcal{U} \oplus \mathcal{U} \mid V, W \text{ indexing spaces in } \mathcal{U}\}.$$

Definition 1.2.15. For two G-prespectra E and F indexed on a G-universe $\mathcal{U}$, the **external smash product** $E \wedge F$ is the G-prespectrum indexed on the G-universe $\mathcal{U} \oplus \mathcal{U}$, induced by the following G-prespectrum indexed on D:

$$(E \wedge F)(V \oplus W) := EV \wedge FW$$

with structure maps

$$\Sigma^{(V' \oplus W')-(V \oplus W)} EV \wedge FW \cong \Sigma^{(V'-V)\oplus(W'-W)} EV \wedge FW$$

$$\cong \Sigma^{V'-V} EV \wedge \Sigma^{W'-W} FW \xrightarrow{\sigma_{V,V'} \wedge \sigma_{W,W'}} EV' \wedge FW'.$$

The external smash product together with the change of universe functor allows us to define a smash product of G-spectra. Let $\mathcal{U}$ be a G-universe and $f : \mathcal{U} \oplus \mathcal{U} \to \mathcal{U}$ a linear isometric isomorphism.

Definition 1.2.16. For two G-spectra E and F indexed on $\mathcal{U}$, the **smash product** $E \wedge F$ is

$$E \wedge F := f_* L(lE \wedge lF).$$

For more details and to see why this definition does not depend on the choice of f in the category $\bar{h}G\mathscr{S}\mathcal{U}$, see [LMS86, p. 72 and Theorem 1.7, p. 61] and [MPC96, Chapter XII.3, p. 115].

Definition 1.2.17. For a family of G-(pre)spectra $\{E_i\}_{i \in I}$ the **wedge product** $\bigvee_{i \in I} E_i$ has spaces

$$(\bigvee_{i \in I} E_i)V := \bigvee_{i \in I} (E_i V)$$

and structure maps

$$\sigma_{V,W} : \quad \Sigma^{W-V}(\bigvee_{i \in I} E_i V) \xrightarrow{\cong} \bigvee_{i \in I} \Sigma^{W-V} E_i V \xrightarrow{\bigvee_{i \in I} \sigma^i_{V,W}} \bigvee_{i \in I} E_i W_i.$$

Definition 1.2.18. Given an indexing space in a G-universe $\mathcal{U}$ the **evaluation functor** $\Omega^\infty_V : G\mathscr{S} \to G\mathscr{T}$ is defined by setting

$$\Omega^\infty_V E := EV$$

for a G-spectrum E and

$$\Omega^\infty_V f := f_V$$

on maps.

Definition 1.2.19. Given a G-space X and an indexing space V, the V**th desuspension prespectrum** $(\{\Sigma^{--V} X\}, \sigma_{-,-})$ has

$$(\Sigma^{--V} X)W := \begin{cases} \Sigma^{W-V} X \text{ for } V \subset W \\ \text{pt otherwise} \end{cases}$$

as spaces and for $W \subset Z$ the identification

$$\sigma_{W,Z} : \ \Sigma^{W-Z} \Sigma^{Z-V} X \xrightarrow{\cong} \Sigma^{W-V} X$$

as structure maps. Suspending maps as well gives a functor

$$G\mathscr{T} \to G\mathscr{P}.$$

Combining this functor with the spectrification we get the **shift desuspension functor**

$$\Sigma_V^\infty - : G\mathscr{T} \to G\mathscr{S},$$

such that $\Sigma_V^\infty X$ is $L(\{\Sigma^{--V} X\})$.

We give a few properties of the shift desuspension functor. (See [LMS86, p. 22 and p. 59].)

Remark 1.2.20. Let $f : \mathcal{U} \to \mathcal{U}'$ be a linear isometric isomorphism between two G-universes as above and let $V \subset \mathcal{U}$ be an indexing space and X a G-space. There is a natural isomorphism

$$f_*(\Sigma_V^\infty X) \cong \Sigma_{fV}^\infty X.$$

Remark 1.2.21. For any indexing space V, the composition $\Omega_V^\infty \circ \Sigma_V^\infty$:

$$G\mathscr{T} \xrightarrow{\Sigma_V^\infty} G\mathcal{S} \xrightarrow{\Omega_V^\infty} G\mathscr{T}$$

is the identity functor.

Remark 1.2.22. The shift desuspension functor Σ_V^∞ is left adjoint to the evaluation functor Ω_V^∞.

Remark 1.2.23. For spaces X and Y there is a natural isomorphism

$$\Sigma_V^\infty(X \wedge Y) \cong (\Sigma_V^\infty X) \wedge Y.$$

Remark 1.2.24. For a space X and isomorphic indexing spaces $V \cong V'$ in $\mathcal{U}$ there is a natural isomorphism

$$\Sigma_V^\infty X \cong \Sigma_{V'}^\infty X.$$

Remark 1.2.25. For a space X and indexing spaces $V \subset W$ in $\mathcal{U}$ there is a natural isomorphism

$$\Sigma_V^\infty X \cong \Sigma_W^\infty \Sigma^{W-V} X.$$

Remark 1.2.26. For spaces X and Y and indexing spaces V and W with $V \cap W = \{0\}$ there is a natural isomorphism

$$\Sigma_{V \oplus W}(X \wedge Y) \cong \Sigma_V^\infty X \wedge \Sigma_W^\infty Y.$$

Lemma 1.2.27. *For a prespectrum E there is a natural isomorphism*

$$LE \cong \operatorname{colim}_V \Sigma_V^\infty EV.$$

Here the colimit is taken over the maps

$$\Sigma_V^\infty EV \xrightarrow{\cong} \Sigma_W^\infty(\Sigma^{W-V} EV) \xrightarrow{\Sigma_W^\infty \sigma_{V,W}} \Sigma_W^\infty EW\ .$$

A proof can be found in [LMS86, p. 25].

Definition 1.2.28. Given a pair of indexing spaces V and W, the **sphere G-spectrum** is

$$S^{W \ominus V} := \Sigma_V^\infty S^W.$$

Definition 1.2.29. For Σ_0^∞ we also write Σ^∞ and for $S^{W\ominus 0}$ we also write $S^W = \Sigma_0^\infty S^W = \Sigma^\infty S^W$, unless there is a possibility of confusing the space S^W with the spectrum S^W. Given a G-(pre)spectrum E the Wth **suspension** of E is

$$\Sigma^W E := E \wedge S^W.$$

Lemma 1.2.30. *Given a G-linear isomorphism*

$$\alpha : V \oplus W' \to V' \oplus W$$

there is an isomorphism of G-spectra:

$$\alpha' : S^{W \ominus V} \to S^{W' \ominus V'}.$$

Proof. Define α' as

$$\Sigma^\infty_V S^W \xrightarrow[\text{Remark 1.2.25}]{\cong} \Sigma^\infty_{V\oplus W'} \Sigma^{W'} S^W \xrightarrow[\text{Remark 1.2.24}]{\cong} \Sigma^\infty_{W'\oplus V} \Sigma^{W'} S^W \xrightarrow[\Sigma^\infty_\alpha \tau]{\cong} \Sigma^\infty_{V'\oplus W} \Sigma^W S^{W'} \xrightarrow[\text{Remark 1.2.25}]{\cong} \Sigma^\infty_{V'} S^{W'},$$

where $\tau : \Sigma^{W'} S^W \to \Sigma^W S^{W'}$ is the transposition isomorphism. □

Definition 1.2.31. For G-spaces X and Y the **stable homotopy classes** of maps $X \to Y$ are

$$\{X, Y\}_G := [\Sigma^\infty X, \Sigma^\infty Y]_G.$$

Definition 1.2.32. We write $\mathscr{I}O(G)$ or $\mathscr{I}O(G,\mathcal{U})$ for the **category of indexing spaces** indexed on a G-universe $\mathcal{U}$. The objects are the indexing spaces in a given G-universe $\mathcal{U}$ and the morphisms between two indexing spaces are the linear isometric isomorphisms between them. Let two such morphisms $V \to W$ be called homotopic if their associated morphisms $S^V \to S^W$ of G-spaces are stably homotopic. The resulting homotopy category is denoted by $h\mathscr{I}O(G)$.

Definition 1.2.33. An **equivariant ring spectrum** is an object E in $\bar{h}G\mathscr{S}$ together with maps $\mu : E \wedge E \to E$ and $\nu : S^0 \to E$ such that the following diagrams commute:

$$\begin{tikzcd} E \wedge E \wedge E \arrow[r, "1\wedge\mu"] \arrow[d, "\mu\wedge 1"] & E \wedge E \arrow[d, "\mu"] \\ E \wedge E \arrow[r, "\mu"] & E \end{tikzcd}$$

and

$$\begin{tikzcd} S^0 \wedge E \arrow[r, "\nu\wedge 1"] \arrow[dr, "\cong"'] & E \wedge E \arrow[d, "\mu"] & E \wedge S^0 \arrow[l, "1\wedge\nu"'] \arrow[dl, "\cong"] \\ & E & \end{tikzcd} .$$

It is called a **commutative** ring spectrum if

$$\begin{tikzcd} E \wedge E \arrow[rr, "\tau", "\cong"'] \arrow[dr, "\mu"'] & & E \wedge E \arrow[dl, "\mu"] \\ & E & \end{tikzcd}$$

commutes. Here τ denotes the transposition isomorphism.

For more details see [MPC96, Chapter III.5, p. 140] and [Swi75, Chapter 13, p. 269].

Equivariant (co)homology theories

A simple way of defining an equivariant homology theory is to take the non-equivariant axiomatic approach, while replacing $\mathscr{T}$ by $G\mathscr{T}$. This is what tom Dieck calls *unstable* equivariant homology theory (compare [tD87, II,(6.7)]).

Definition 1.2.34. A $\mathbb{Z}$-graded equivariant homology theory with respect to the group G is a homology theory in the sense of [May99, Chapter 14.4, p. 108], replacing the homotopy category of based spaces by the homotopy category of based G-spaces.

Definition 1.2.35. A $\mathbb{Z}$-graded equivariant cohomology theory with respect to the group G is a cohomology theory in the sense of [May99, Chapter 19.2, p. 144], replacing the homotopy category of based spaces by the homotopy category of based G-spaces.

Compare also Bredon's notion of equivariant (co)homology [Bre67].

The following definitions are from [MPC96, Chapter XIII, p. 130], although only the co̲homological part is spelled out there.

Definition 1.2.36. A **homology theory graded over a G-universe $\mathcal{U}$** is a functor

$$E^G_-(-) : h\mathscr{I}O(G)^{op} \times (\overline{h}G\mathscr{T}) \longrightarrow \mathcal{A}b$$

together with suspension isomorphisms

$$\sigma^W : E^G_V(X) \longrightarrow E^G_{V\oplus W}(\Sigma^W X),$$

which are natural in V and in X, such that the following axioms are satisfied.

1. For each indexing space V, the functor E^G_V sends wedges to sums and is exact on cofiber sequences.
2. For a map $\alpha : W \longrightarrow W'$ between indexing spaces, the following diagram commutes:

$$\begin{tikzcd}
E^G_V(X) \arrow[rr, "\sigma^W"] \arrow[d, "\sigma^{W'}"] & & E^G_{V\oplus W}(\Sigma^W X) \arrow[d, "E^G_{\mathrm{id}\,\oplus\,\mathrm{id}}(\Sigma^\alpha\,\mathrm{id})"] \\
E^G_{V\oplus W'}(\Sigma^{W'} X) \arrow[rr, "E^G_{\mathrm{id}\,\oplus\alpha}(\mathrm{id})"] & & E^G_{V\oplus W}(\Sigma^{W'} X).
\end{tikzcd}$$

3. $\sigma^0 = \mathrm{id}$ and the suspension isomorphisms are transitive in the following sense:

$$\begin{tikzcd}
E^G_V(X) \arrow[rr, "\sigma^W"] \arrow[dr, "\sigma^{W\oplus W'}"'] & & E^G_{V\oplus W}(\Sigma^W X) \arrow[dl, "\sigma^{W'}"] \\
& E^G_{V\oplus W\oplus W'}(\Sigma^{W\oplus W'} X). &
\end{tikzcd}$$

Definition 1.2.37 (compare [Cos96, Definition 1.1])**.** A **cohomology theory graded over a G-universe $\mathcal{U}$** is a functor

$$E^-_G(-) : h\mathscr{I}O(G) \times \overline{h}G\mathscr{T}^{op} \longrightarrow \mathcal{A}b$$

together with suspension isomorphisms

$$\sigma^W : E^V_G(X) \longrightarrow E^{V\oplus W}_G(\Sigma^W X),$$

which are natural in V and in X, such that the following axioms are satisfied.

1. For each indexing space V, the functor E^V_G sends wedges to products and is exact on cofiber sequences.
2. For a map $\alpha : W \longrightarrow W'$ between indexing spaces, the following diagram commutes:

$$\begin{tikzcd}
E^V_G(X) \arrow[rr, "\sigma^W"] \arrow[d, "\sigma^{W'}"] & & E^{V\oplus W}_G(\Sigma^W X) \arrow[d, "E^{\mathrm{id}\,\oplus\alpha}_G(\mathrm{id})"] \\
E^{V\oplus W'}_G(\Sigma^{W'} X) \arrow[rr, "E^{\mathrm{id}\,\oplus\,\mathrm{id}}_G(\Sigma^\alpha\,\mathrm{id})"] & & E^{V\oplus W'}_G(\Sigma^W X).
\end{tikzcd}$$

3. $\sigma^0 = \mathrm{id}$ and the suspension isomorphisms are transitive in the following sense:

$$\begin{tikzcd} E^V_G(X) \arrow[rr, "\sigma^W"] \arrow[dr, "\sigma^{W\oplus W'}"'] & & E^{V\oplus W}_G(\Sigma^W X) \arrow[dl, "\sigma^{W'}"] \\ & E^{V\oplus W\oplus W'}_G(\Sigma^{W\oplus W'} X). & \end{tikzcd}$$

Definition 1.2.38. Analogous to definition 1.2.36 we define a **homology theory indexed on a G-universe $\mathcal{U}$ on the category of G-spectra**, replacing $\overline{h}G\mathscr{T}$ by $\overline{h}G\mathscr{S}$. Analogous to definition 1.2.37 we define a **cohomology theory indexed on a G-universe $\mathcal{U}$ on the category of G-spectra**, replacing $\overline{h}G\mathscr{T}$ by $\overline{h}G\mathscr{S}$.

Remark 1.2.39. Given a homology theory graded over a G-universe $\mathcal{U}$ and an indexing space V with trivial fixed set, it follows from the first axiom of Definition 1.2.36 that E^G_{V+n} together with the suspension isomorphisms σ^1 gives a $\mathbb{Z}$-graded equivariant homology theory. Analogously we obtain an equivariant $\mathbb{Z}$-graded cohomology theory from a cohomology theory graded over $\mathcal{U}$.

Lemma 1.2.40. *An Ω-G-spectrum E indexed on a G-universe $\mathcal{U}$ defines a functor*

$$h\mathscr{I}O(G,\mathcal{U}) \to \overline{h}G\mathscr{T}$$

that is the evaluation EV on objects V in $h\mathscr{I}O(G,\mathcal{U})$.

Proof. See [MPC96, Chapter XIII, Lemma 2, p. 132] □

Proposition 1.2.41. *An Ω-G-spectrum E indexed on a G-universe $\mathcal{U}$ represents a homology theory graded over $\mathcal{U}$ on G-spaces.*

Proof. For an indexing space $V \subset \mathcal{U}$, define

$$E^G_V(X) := [S^V, E \wedge X]_G.$$

For a map of G-spaces $\phi : X \to Y$ we obtain a map of Ω-G-spectra

$$1 \wedge \phi : E \wedge X \to E \wedge Y$$

and

$$E^G_V(f):\ [S^V, E \wedge X]_G \xrightarrow{1\wedge\phi\circ -} [S^V, E \wedge Y]_G$$

is defined by post-composing with $1 \wedge \phi$. For a map $\alpha : V \to W$ in $h\mathscr{I}O(G)$ we obtain a map

$$S^\alpha : S^V \to S^W$$

and

$$E^G_\alpha(X):\ [S^W, E \wedge X]_G \xrightarrow{-\circ S^\alpha} [S^V, E \wedge Y]_G$$

is defined by pre-composing with S^α.

We claim this gives a functor

$$E^G_-(-) : (h\mathscr{I}O(G))^{op} \times \overline{h}G\mathscr{I} \to \mathcal{A}b$$

satisfying the axioms of a homology theory (Definition 1.2.36). Axiom 1 is satisfied just as in the non-equivariant case; see [Swi75, 8.33] for example. The commutativity of the diagrams of Axioms 2 and 3 follows immediately. □

Proposition 1.2.42. *An Ω-G-spectrum E indexed on a G-universe $\mathcal{U}$ represents a cohomology theory graded over $\mathcal{U}$ on G-spaces.*

Proof. For an indexing space $V \subset \mathcal{U}$, define

$$E^V_G(X) := [X, EV]_G.$$

For a map of G-spaces $\phi : X \to Y$, define

$$E^V_G(\phi) := [\phi, EV]_G : [Y, EV]_G \to [X, EV]_G,$$

where $[\phi, EV]_G$ denotes composing with ϕ. Similarly for a map $\alpha : V \to W$, define

$$E^\alpha_G(X) := [X, E\alpha]_G : [X, EV]_G \to [X, EW]_G,$$

where $[X, E\alpha]_G$ denotes composing with $E\alpha$, the map obtained when we view E as evaluation functor (see 1.2.40). We claim that this gives a functor

$$E^-_G(-) : h\mathscr{I}O(G) \times \overline{h}G\mathscr{T}^{op} \longrightarrow \mathcal{A}b,$$

satisfying the axioms of a cohomology theory (definition 1.2.37). Details can be found in [MPC96, Chapter XIII.2, p. 132]. □

Proposition 1.2.43. *A G-spectrum E indexed on a G-universe $\mathcal{U}$ represents a homology theory on G-spectra graded over $\mathcal{U}$.*

Proof. The proof is formally identical to the proof of Proposition 1.2.41. □

Proposition 1.2.44. *A G-spectrum E indexed on a G-universe $\mathcal{U}$ represents a cohomology theory on G-spectra graded over $\mathcal{U}$.*

Proof. The proof is formally identical to the proof of Proposition 1.2.42. □

Notice that for a G-spectrum E and our associated (co)homology theories we have

$$E^V_G(X) = E^V_G(\Sigma^\infty X)$$

and

$$E^G_V(X) = E^G_V(\Sigma^\infty X).$$

For an equivariant ring spectrum we can define an external product on the associated equivariant homology theory indexed on $\mathcal{U}$:

$$E^G_V(X) \otimes E^G_W(Y) \to E^G_{V\oplus W}(X \wedge Y).$$

We give a sketch of how it is defined. A complete treatment, also including other products on (co)homology can be found in [MPC96, p. 139f] and [LMS86, III §3].

We have a map

$$E \wedge X \wedge E \wedge Y \xrightarrow{\mathrm{id}\wedge\tau\wedge\mathrm{id}} E \wedge E \wedge X \wedge Y$$

where τ denotes the transposition and we have the structure map of the ring spectrum E

$$\mu : E \wedge E \to E.$$

Given two elements

$$f \in E^G_V(X) \text{ and } g \in E^G_W(Y)$$

represented by

$$f \in [S^V, E \wedge X]_G \text{ and } g \in [S^W, E \wedge Y]_G,$$

we assign a map

$$S^{W\oplus V} \cong S^V \wedge S^W \xrightarrow{f\wedge g} E \wedge X \wedge E \wedge Y \xrightarrow{\mathrm{id}\wedge\tau\wedge\mathrm{id}} E \wedge E \wedge X \wedge Y \xrightarrow{\mu\wedge\mathrm{id}} E \wedge X \wedge Y.$$

This assignment gives rise to a multiplication

$$E^G_*(X) \otimes E^G_*(Y) \to E^G_*(X \wedge Y).$$

Taking $X = S^0$ we see that $E^G_*(Y)$ carries an E^G_*-module structure.

Remark 1.2.45. Converse statements to the preceding propositions can be found in [MPC96, Chapter XIII, Section 3]. Brown's representation theorem applies and we will quote a corollary without quoting the proof.

Proposition 1.2.46 (see [MPC96, Chapter XIII, Corollary 3.2])**.** *A cohomology theory E_G^* indexed over a G-universe $\mathcal{U}$ is represented by an Ω-G-prespectrum indexed on $\mathcal{U}$.*

The real representation ring

Definition 1.2.47. Given a real G-universe $\mathcal{U}$ the **real representation group**

$$RO(G;\mathcal{U})$$

is the set of equivalence classes of formal differences $V \ominus W$, where V and W are indexing spaces in $\mathcal{U}$ and $V \ominus W$ is equivalent to $V' \ominus W'$ if there is a G-linear isometric isomorphism

$$\alpha : V \oplus W' \to V' \oplus W.$$

The virtual dimension of an element $Z = V \ominus W$ is $|Z| := |V| - |W|$.

Remark 1.2.48. The group structure on $RO(G;\mathcal{U})$ is given by taking the direct sum on representatives. If $\mathcal{U}$ is a complete G-universe or, more generally, if the tensor product of two given indexing spaces is isomorphic to another indexing space, then $RO(G;\mathcal{U})$ can be given a commutative ring structure by taking the tensor product on representatives.

Definition 1.2.49. For a group G and a complete real G-universe $\mathcal{U}$ we choose a set J of real G-representations containing exactly one representative of every isomorphism class of an irreducible non-trivial representation. The free Abelian group $\mathbb{Z}J$ can be considered to be an additive subgroup of $RO(G)$. We set

$$AO_*(G) := \mathbb{Z}[\mathbb{Z}J].$$

This is a graded ring; the grading is induced by the virtual dimension of elements in $\mathbb{Z}J \subset RO(G)$.

Remark 1.2.50. We have an isomorphism

$$AO_*(G) \cong \mathbb{Z}[e_V, e_V^{-1}]_{V\in J},$$

for indeterminates e_V and e_V^{-1} with the obvious relations, which is induced by

$$\mathbb{Z}J \ni \sum_{V\in J} \alpha_V V \mapsto \prod_{V\in J} e_V^{-\alpha_V}.$$

The e_V's are not to be confused with the Euler classes of Definition 1.3.22. Later in Proposition 3.1.10 we will explain why we choose to name them e_V and e_V^{-1}.

Definition 1.2.51. Given a cohomology theory E (graded on a G-universe $\mathcal{U}$) and a pair of indexing spaces V and W we define

$$E_G^{V\ominus W}(X) := E_G^V(\Sigma^W X).$$

Given a G-linear isometric isomorphism

$$\alpha : V \oplus W' \to V' \oplus W$$

we obtain an isomorphism

$$E_G^{V\ominus W}(X) \to E_G^{V'\ominus W'}(X)$$

in the following way:

$$E_G^V(\Sigma^W X) \xrightarrow{\sigma^{W'}} E_G^{V\oplus W'}(\Sigma^{W\oplus W'} X) \xrightarrow{E_G^\alpha(\Sigma^\tau \mathrm{id})} E_G^{V'\oplus W}(\Sigma^{W'\oplus W} X) \xrightarrow{(\sigma^W)^{-1}} E_G^{V'}(\Sigma^{W'} X).$$

Here $\tau : W' \oplus W \to W \oplus W'$, which appears in the middle map, denotes the transposition isomorphism.

We choose a system of representatives of $RO(G;\mathcal{U})$ and also a system of unique isomorphisms within each isomorphism class and define

$$E^a_G(X) := E^{V\ominus W}_G(X)$$

for an element a in $RO(G;\mathcal{U})$, which is represented by $V \ominus W$. This is then well-defined up to unique isomorphism. Notice that in the above definitions the cohomology theories can be viewed as cohomology theories on G-spaces or on G-spectra.

We give an analogous definition for a homology theory:

Definition 1.2.52. Given a homology theory E (graded on a G-universe $\mathcal{U}$) and a pair of indexing spaces V and W we define

$$E^G_{V\ominus W}(X) = E^G_V(\Sigma^W X).$$

Given a G-linear isometric isomorphism $\alpha : V' \oplus W \to V \oplus W'$ we obtain an isomorphism

$$E^G_{V\ominus W}(X) \to E_{V'\ominus W'}(X)$$

in the following way:

$$E^G_V(\Sigma^W X) \xrightarrow{\sigma^{W'}} E^G_{V\oplus W'}(\Sigma^{W\oplus W'} X) \xrightarrow{E^G_\alpha(\Sigma^\tau \,\mathrm{id})} E^{V'\oplus W}_G(\Sigma^{W'\oplus W} X) \xrightarrow{(\sigma^W)^{-1}} E^{V'}_G(\Sigma^{W'} X).$$

Here $\tau : W \oplus W' \to W' \oplus W$, which appears in the middle map, denotes the transposition isomorphism. As in the cohomological case we choose a set of representatives of $RO(G;\mathcal{U})$ and a system of unique isomorphisms and define

$$E^G_a(X) := E_{V\ominus W}(X)$$

for an element a in $RO(G;\mathcal{U})$ represented by $V \ominus W$. These definitions are meant for (co)homology theories on G-spectra as well as on G-spaces. Similar to the cohomological case we get isomorphisms for equivalent formal differences of representations and we adopt the analogous convention we established for cohomology.

Notice that for a cohomology theory on G-spectra represented by a G-spectrum E we have

$$E^{V\ominus W}_G(D) = E^V_G(\Sigma^W D) = [\Sigma^\infty_V \Sigma^W D, E]_G = [S^{W\ominus V} \wedge D, E]_G$$

and for a homology theory on G-spectra represented by a G-spectrum E we have

$$\begin{aligned}
E^G_{V\ominus W}(D) &= E^G_V(\Sigma^W D) && \text{by 1.2.52}\\
&= [\Sigma^\infty S^V, \Sigma^W E \wedge D]_G && \text{by 1.2.41}\\
&\cong [\Sigma^\infty_W \Sigma^W S^V, \Sigma^W E \wedge D]_G && \text{by 1.2.25}\\
&= [\Sigma^W \Sigma^\infty_W S^V, \Sigma^W E \wedge D]_G && \text{by 1.2.23}\\
&= [\Sigma^\infty_W S^V, E \wedge D]_G && \text{by supsension isomorphism}\\
&= [S^{V\ominus W}, E \wedge D]_G. && \text{by 1.2.28}
\end{aligned}$$

The complex representation ring

Definition 1.2.53. Given a complex G-universe $\mathcal{U}$ the **complex representation group** $R(G;\mathcal{U})$ or $R(G)$ is the set of equivalence classes of formal differences $V \ominus W$, where V and W are indexing spaces in $\mathcal{U}$ and $V \ominus W$ is equivalent to $V' \ominus W'$ if there is a G-linear isometric isomorphism

$$\alpha : V \oplus W' \to V' \oplus W.$$

As in the real case we have a group structure on $R(G)$. We follow [Han05, p. 683] and [tD70, §2].

Definition 1.2.54. Let J be a system of representatives for the isomorphism classes of non-trivial irreducible complex representations of G.

Notice that $\mathbb{Z}J$, the free Abelian group on J, can be viewed as an additive subgroup of $R(G)$.

Definition 1.2.55. Given a group G and J as above we set

$$A_*(G) := \mathbb{Z}[\mathbb{Z}J].$$

This is a graded ring:

$$A_*(G) = \bigoplus_{n=-\infty}^{\infty} A_n(G)$$

with

$$A_n(G) := \left\{ \sum_{i\in\mathbb{Z}} \beta_i \big(\sum_{V\in\mathbb{Z}J} \alpha_V^i V \big) \in A_*(G) \,\middle|\, \beta_i, \alpha_V^i \in \mathbb{Z}, \sum \alpha_V^i |V| = n \right\},$$

where $|V|$ denotes the real dimension of V. Analogous to Remark 1.2.50 we have an isomorphism

$$A_*(G) \cong \mathbb{Z}[e_V, e_V^{-1}]_{V\in J}$$

and again e_V and e_V^{-1} are not the Euler classes (see Definition 1.3.22), but just indeterminates.

Geometric fixed point spectra

Let $\mathcal{U}$ be a G-universe and let $\{K_n\}_{n\geq 0}$ be a sequence of indexing spaces in $\mathcal{U}$ with $K_n \subset K_{n+1}$, such that every irreducible representation of G is contained arbitrarily often in K_N for large N and such that $K_{n+1} - K_n$ contains exactly one copy of the trivial representation, i.e. $|(K_{n+1} - K_n)^G| = 1$. For finite G we can choose K_n as the n-fold product of the regular representation.

Definition 1.2.56. Let E be a G-prespectrum indexed on a G-universe $\mathcal{U}$ and let $\{K_n\}_{n\geq 0}$ be a sequence of indexing spaces as above. The **geometric fixed point spectrum** $\Phi^G E$ is the spectrification of the (non-equivariant) prespectrum $\phi^G E$ defined as follows: For $n \in \mathbb{N}$

$$(\phi^G E)_n := (E_{K_n})^G$$

and the adjoint of the structure map $(\phi^G E)_n \to \Omega(\phi^G E)_{n+1}$ is defined as

$$\begin{aligned}(E_{K_n})^G &\to (\Omega^{K_{n+1}-K_n} E_{K_{n+1}})^G \\ &\to \Omega^{(K_{n+1}-K_n)^G} (E_{K_{n+1}})^G = \Omega(E_{K_{n+1}})^G,\end{aligned}$$

where the first map is the restriction of the structure map

$$\sigma_{K_n,K_{n+1}} : (E_{K_n}) \to (\Omega^{K_{n+1}-K_n} E_{K_{n+1}})$$

of E and the second map is the restriction to fixed sets of the loop space.

The following properties can be found in [MPC96, Chapter XVI.3] (also compare [LMS86, Chapter II §9 and Chapter I §3]).

Remark 1.2.57. The spectrum $\Phi^G E$ does not depend on the choice of $\{K_n\}_{n\geq 0}$.

Taking the geometric fixed point spectrum can be extended to a functor

$$\Phi^G : G\mathscr{S}\mathcal{U} \to \mathscr{S}.$$

This functor has the following properties.

Remark 1.2.58. For a G-space X we have

$$\Sigma^\infty(X^G) \cong \Phi^G(\Sigma^\infty X).$$

Remark 1.2.59. Taking the geometric fixed point spectrum is compatible with the smash product. For G-spectra E and E' we have

$$\Phi^G(E) \wedge \Phi^G(E') \cong \Phi^G(E \wedge E').$$

1.3 Equivariant homotopic (co)bordism and Thom spectra

Definition 1.3.1. A G-bundle is a bundle (E, B, π) such that E and B are G-spaces and $\pi : E \to B$ is equivariant.

See [tD87, Section I.8] for details.

Definition 1.3.2. A real (resp. complex) **G-vector bundle** is a real (resp. complex) vector bundle $p : E \to B$ such that E and B are G-spaces and p is equivariant and for each $b \in B, g \in G$, the left translation

$$\begin{aligned} L_g : E_b &\to E_{gb} \\ x &\mapsto gx \end{aligned}$$

is linear. A map of G-vector bundles is an equivariant map of the underlying vector bundles.

See [tD87, Chapter 9] and [MPC96, Chapter XVI.1] for details.

Definition 1.3.3. The category of G-vector bundles and maps between them is denoted by $G\mathscr{V}$.

Definition 1.3.4. Given a G-vector bundle $p : E \to B$ we apply one-point compactification to each fiber of p and obtain a new G-bundle S^E with fibers based spheres S^n (with base points ∞). The base points give a section $B \to S^E$ and we define the **Thom space** of p as the quotient

$$T(p) := S^E/B.$$

Remark 1.3.5. The above construction of taking Thom spaces is functorial, i.e. can be extended to maps of vector bundles to give a functor

$$T : G\mathscr{V} \to G\mathcal{T}$$

from the category of G-vector bundles to the category of G-spaces.

The real equivariant Thom spectrum

Classifying G-vector bundles is analogous to the non-equivariant case. We give definitions to fix notation and also define Thom spectra. More details can be found in [LMS86, Chapter X]. Let V be an indexing space in a complete real G-universe $\mathcal{U}$ with $|V| = n$.

Definition 1.3.6. Define the **classifying space** $BO(|V|, V \oplus \mathcal{U})$ to be the set of real n-planes in $V \oplus \mathcal{U}$. If the universe is clear from context, the notation $BO^G(n)$ is also used for $BO(n, V \oplus \mathcal{U})$.

Remark 1.3.7. The set $BO(|V|, V \oplus \mathcal{U})$ can be topologized (in the same way as the grassmannians) and we can give it the structure of a G-space by using the G-action on $V \oplus \mathcal{U}$ to send a n-plane in $V \oplus \mathcal{U}$ to another n-plane in $V \oplus \mathcal{U}$ and take $V \subset V \oplus \mathcal{U}$ as a base point.

Definition 1.3.8. The **tautological bundle** $EO(|V|, V \oplus \mathcal{U})$ over $BO(|V|, V \oplus \mathcal{U})$ is the set

$$EO(|V|, V \oplus \mathcal{U}) := \{(p, W) \in (V \oplus \mathcal{U}) \times BO(|V|, V \oplus \mathcal{U}) \,|\, p \in W\},$$

topologized as a subset of $(V \oplus \mathcal{U}) \times BO(|V|, V \oplus \mathcal{U})$ with the restricted G-action and $(0, V)$ as a base point together with the projection on the second variable:

$$\begin{aligned} \pi(V) : EO(|V|, V \oplus \mathcal{U}) &\to BO(|V|, V \oplus \mathcal{U}) \\ (p, W) &\mapsto W. \end{aligned}$$

If the universe is clear from context, the notation $EO^G(V)$ is also used for the tautological bundle.

Remark 1.3.9. The real tautological bundle is a real n-plane G-vector bundle.

Definition 1.3.10. The real equivariant **Thom prespectrum** TO_G has as Vth space $TO_G(V)$, the Thom space of $\pi(V)$, the tautological bundle over $BO(|V|, V \oplus \mathcal{U})$. For $V \subset W$ the pullback of $\pi(W)$ under the inclusion

$$BO(|V|, V \oplus \mathcal{U}) \to BO(|W|, W \oplus \mathcal{U})$$

is the direct sum of the trivial G-vector bundle with fiber $W - V$ and $\pi(V)$, so that its Thom space is (canonically homeomorphic to) $\Sigma^{W-V} TO_G(V)$. Passing to Thom spaces then gives a map

$$\sigma_{V,W} : \Sigma^{W-V} TO_G(V) \to TO_G(W),$$

which we define to be the structure maps of TO_G. We sometimes write TO^G instead of TO_G.

Definition 1.3.11. The real equivariant **Thom spectrum** MO_G is the spectrification of the real equivariant Thom prespectrum:

$$MO_G := LTO_G.$$

We sometimes write MO^G instead of MO_G.

Definition 1.3.12. The homology theory associated to the real equivariant Thom spectrum MO^G is called real equivariant homotopic bordism.

Definition 1.3.13. The cohomology theory associated to the real equivariant Thom spectrum MO_G is called real equivariant homotopic cobordism.

Notice that with Lemma 1.2.27 we have

$$MO_G \cong \mathrm{colim}_W \, \Sigma^\infty_W TO_G W.$$

Lemma 1.3.14. *We have the following description of the coefficients of real equivariant homotopic bordism in positive degrees:*

$$MO^G_V = MO^G_V(S^0) \cong \mathrm{colim}_W [S^{V \oplus W}, TO_G(W)]_G.$$

Proof.

$$\begin{aligned}
MO^G_V(S^0) &\cong [S^{V \ominus 0}, LTO_G \wedge S^0] \cong [\Sigma^\infty_0 S^V, \mathrm{colim}_W \, \Sigma^\infty_W TO_G(W)]_G \\
&\cong \mathrm{colim}_W [\Sigma^\infty_0 S^V, \Sigma^\infty_W TO_G(W)]_G \\
&\cong \mathrm{colim}_W [\Sigma^\infty_W \Sigma^W S^V, \Sigma^\infty_W TO_G(W)]_G \\
&\cong \mathrm{colim}_W [S^{W \oplus V}, \Omega^\infty_W \Sigma^\infty_W TO_G(W)]_G \\
&\cong \mathrm{colim}_W [S^{V \oplus W}, TO_G(W)]_G.
\end{aligned}$$

□

Lemma 1.3.15. *We have the following description of the coefficients of real equivariant homotopical bordism in negative degrees:*

$$MO^G_{-V} = MO^G_{-V}(S^0) \cong \mathrm{colim}_W [S^W, TO_G(V \oplus W)]_G.$$

Proof.

$$\begin{aligned}
MO^G_{-V}(S^0) &\cong [S^{0 \ominus V} \wedge S^0, LTO_G]_G \\
&\cong [\Sigma^\infty_V S^0, \mathrm{colim}_W \, \Sigma^\infty_W TO_G(W)]_G \\
&\cong \mathrm{colim}_W [\Sigma^\infty_V S^0, \Sigma^\infty_W TO_G(W)]_G \\
&\cong \mathrm{colim}_{V \subset W} [\Sigma^\infty_V S^0, \Sigma^\infty_W TO_G(W)]_G \\
&\cong \mathrm{colim}_{Z = V \oplus W} [\Sigma^\infty_V S^0, \Sigma^\infty_Z TO_G(Z)]_G \\
&\cong \mathrm{colim}_{Z = V \oplus W} [\Sigma^\infty_Z \Sigma^{Z-V} S^0, \Sigma^\infty_Z TO_G(Z)]_G \\
&\cong \mathrm{colim}_{Z = V \oplus W} [S^W, TO_G(V \oplus W)]_G.
\end{aligned}$$

□

Next we want to describe how a ring structure on MO_G can be defined. Classifying the product of tautological bundles, we obtain a map

$$p_{V,W} : BO(|V|, V \oplus \mathcal{U}) \times BO(|W|, W \oplus \mathcal{U}) \to BO(|V| + |W|, V \oplus W \oplus \mathcal{U})$$

such that for the pullback of the tautological bundle we have

$$p^*_{V,W,}(EO(|V| + |W|, V \oplus W \oplus \mathcal{U})) \cong EO(|V|, V \oplus \mathcal{U}) \times EO(|W|, W \oplus \mathcal{U}).$$

Then

$$\begin{array}{ccc} EO(|V|, V \oplus \mathcal{U}) \times EO(|W|, W \oplus \mathcal{U}) & \xrightarrow{\tilde{p}_{V,W}} & EO(|V| + |W|, V \oplus W \oplus \mathcal{U}) \\ \downarrow{\scriptstyle \pi(V)\times\pi(W)} & & \downarrow{\scriptstyle \pi(V\oplus W)} \\ BO(|V|, V \oplus \mathcal{U}) \times BO(|W|, W \oplus \mathcal{U}) & \xrightarrow{p_{V,W}} & BO(|V| + |W|, V \oplus W \oplus \mathcal{U}) \end{array}$$

commutes and passing to Thom spaces gives rise to a map

$$TO_G(V) \wedge TO_G(W) \to TO_G(V \oplus W).$$

Remark 1.3.16. The above construction defines a ring spectrum structure on MO_G:

$$\mu : MO_G \wedge MO_G \to MO_G.$$

Non-equivariantly we will use the following H-space, which is defined in a similar way.

Definition 1.3.17. For $n \in \mathbb{N}$ let $BO(n)$ be the set of n-planes in $\mathbb{R}^\infty$ topologized in the usual way (see for example [May99, p. 184]).

The space $BO(n)$ together with the universal bundle $\xi_n : E_n \to BO(n)$ is a classifying space for real n-dimensional vector bundles. We have a map

$$i_n : BO(n) \to BO(n+1),$$

which is characterized up to homotopy by

$$i_n^*(\xi_{n+1}) \cong \xi_n \oplus \varepsilon,$$

where ε is the trivial one-dimensional bundle. We set

$$BO := \operatorname{colim}_n BO(n).$$

Remark 1.3.18. The space BO has an H-space structure

$$H : BO \times BO \to BO,$$

which is induced by maps

$$p_{m,n,} : BO(m) \times BO(n) \to BO(m+n),$$

characterized up to homotopy by

$$p^*_{m+n}(\xi_{m+n}) \cong \xi_m \oplus \xi_n.$$

We give BO a base point $*$ such that $H(*, *) = *$.

Compare [tD70, §2] for the complex analogue BU.

Periodicity

Remark 1.3.19. The definition of the Thom space $TO_G(V)$ only depends on the dimension of V. This can be seen from the definition of the Thom prespectrum (Definition 1.3.10): already the isomorphism class of $EO^G(V)$ depends only on the dimension of V, see Definitions 1.3.6 and 1.3.8.

Lemma 1.3.20. *For every indexing space V, we have an equivalence of G-spectra*

$$\Sigma^V MO_G \simeq \Sigma^{|V|} MO_G$$

and an equivalence

$$\Sigma^{V-|V|} MO_G \simeq MO_G.$$

Proof. A proof can be found in [MPC96, Chapter XV.2, p. 157]. The equivalence is constructed as follows. We have a map $S^V \to TO_G(|V|)$, which comes from the classifying map of V, regarding V as a bundle $V \to *$ over a point. Adjoint to this map, we get a map of G-spectra

$$\Sigma_n^\infty S^V \to MO_G$$

and this gives the desired map

$$\Sigma_n^\infty S^V \wedge MO_G \longrightarrow MO_G \wedge MO_G \xrightarrow{\mu} MO_G.$$

□

Lemma 1.3.21. *For every space X and every pair of indexing spaces V and W there is an isomorphism*

$$MO^G_{V\ominus W}(X) \cong MO^G_{|V|-|W|}(X).$$

Proof.

$$\begin{aligned} MO^G_{V\ominus W}(X) &\cong MO^G_V(\Sigma^W X) \\ &\cong MO^G_V(\Sigma^{|W|} X) \\ &\cong \operatorname{colim}_Z [S^Z, TO_G(V\oplus Z)\wedge \Sigma^{|W|} X]_G \\ &\cong \operatorname{colim}_Z [S^Z, TO_G(|V|\oplus Z)\wedge \Sigma^{|W|} X]_G \\ &\cong MO^G_{|V|}(\Sigma^{|W|} X) \end{aligned}$$

□

Definition 1.3.22. For an indexing space V, the inclusion of the base point (twice)

$$\overline{\phi} : 0 \to V \to BO(|V|, V \oplus \mathcal{U})$$

gives a map

$$\phi : S^0 \to S^V \to TO_G(V),$$

by passing to Thom spaces. The **Euler class** $e_V \in MO^G_{-V}$ is the image of the homotopy class of ϕ in the colimit $MO^G_{-V} \cong \operatorname{colim}_W [S^W, TO(V \oplus W)]_G$.

The complex equivariant Thom spectrum

The definition of the complex equivariant Thom spectrum parallels the real case. Let V be an indexing space in a complete complex G-universe $\mathcal{U}$ with $|V| = n$.

Definition 1.3.23. Define $BU(|V|, V \oplus \mathcal{U})$ to be the set of complex n-planes in $V \oplus \mathcal{U}$.

Remark 1.3.24. The set $BU(|V|, V \oplus \mathcal{U})$ can be topologized (in the same way as the grassmannians) and we can give it the structure of a G-space by using the G-action on $V \oplus \mathcal{U}$ to send an n-plane in $V \oplus \mathcal{U}$ to another n-plane in $V \oplus \mathcal{U}$ and take $V \subset V \oplus \mathcal{U}$ as a base point.

Definition 1.3.25. The **tautological bundle** $EU(|V|, V \oplus \mathcal{U})$ over $BU(|V|, V \oplus \mathcal{U})$ is the set

$$EU(|V|, V \oplus \mathcal{U}) := \{(p, W) \in (V \oplus \mathcal{U}) \times BU(|V|, V \oplus \mathcal{U}) \,|\, p \in W\},$$

topologized as subset of $(V \oplus \mathcal{U}) \times BU(|V|, V \oplus \mathcal{U})$ with restricted G-action and $(0, V)$ as a base point together with the projection on the second variable:

$$\begin{aligned} \pi(V) : EU(|V|, V \oplus \mathcal{U}) &\to BU(|V|, V \oplus \mathcal{U}) \\ (p, W) &\mapsto W. \end{aligned}$$

Remark 1.3.26. The complex tautological bundle is a complex n-plane G-vector bundle.

Definition 1.3.27. The complex equivariant **Thom prespectrum** TU_G has as Vth space the Thom space of $\pi(V)$, the tautological bundle over $BU(|V|, V \oplus \mathcal{U})$. For $V \subset W$ the pullback of $\pi(W)$ under the inclusion

$$BU(|V|, V \oplus \mathcal{U}) \to BU(|W|, W \oplus \mathcal{U})$$

is the direct sum of the trivial G-vector bundle with fiber $W - V$ and $\pi(V)$, so that its Thom space is (canonically homeomorphic to) $\Sigma^{W-V} TU_G(V)$. The map induces an inclusion

$$\sigma_{V,W} : \Sigma^{W-V} TU_G(V) \to TU_G(W),$$

which we define to be the structure maps of TU_G. We sometimes write TU^G instead of TU_G.

Definition 1.3.28. The complex equivariant **Thom spectrum** MU_G is the spectrification of the complex equivariant Thom prespectrum:

$$MU_G := LTU_G.$$

We sometimes write MU^G instead of MU_G.

Definition 1.3.29. The homology theory associated to the complex equivariant Thom spectrum MU^G is called complex equivariant homotopic bordism.

Definition 1.3.30. The cohomology theory associated to the complex equivariant Thom spectrum MU_G is called complex equivariant homotopic cobordism.

Remark 1.3.31 (compare [Han05, Section 2]). The homology theory MU^G_* of Definition 1.3.29 specializes to a $\mathbb{Z}$-graded homology theory that can be redefined as follows. For a G-CW-complex X set

$$MU^G_{2k}(X) := \operatorname{colim}_W [S^W, TU^G_{|W|-k} \wedge X]^G,$$

where the colimit is taken over complex representations on a complete G-universe. For odd degrees we set

$$MU^G_{2k-1}(X) := MU^G_{2k}(S^1 \wedge X),$$

where S^1 carries the trivial G-action.

From now on we will exclusively mean the $\mathbb{Z}$-graded homology theory just defined when we talk about $MU^G_*(-)$.

1.4 Equivariant geometric (co)bordism

Real equivariant geometric (co)bordism

Definition 1.4.1. A **n-G-manifold** is an n-manifold M together with a smooth G-action: $G \times M \to M$. If there is no ambiguity about the group or the dimension we will sometimes write manifold instead of n-G-manifold.

In the following definitions, we only consider compact manifolds.

Definition 1.4.2. Two closed n-G-manifolds M_1 and M_2 are **cobordant** if there is an $(n+1)$-G-manifold W such that ∂W is equivariantly diffeomorphic to $M_1 \amalg M_2$.

Remark 1.4.3. "Cobordant" gives an equivalence relation on G-manifolds.

Definition 1.4.4. By $\mathfrak{N}_n^G$ we denote the set of cobordism classes of n-G-manifolds. If M is a manifold, the element it represents in $\mathfrak{N}_n^G$ is denoted by $[M]$.

Remark 1.4.5. With addition induced by taking the disjoint union of representatives, $\mathfrak{N}_n^G$ is an Abelian group. Since for any manifold M,

$$[M \amalg M] = 0,$$

$\mathfrak{N}_n^G$ is a $\mathbb{Z}/2$ vector space.

Definition 1.4.6. We set

$$\mathfrak{N}_*^G := \bigoplus_{i=0}^{\infty} \mathfrak{N}_i^G.$$

Remark 1.4.7. With multiplication induced by the product of representatives, $\mathfrak{N}_*^G$ is a graded $\mathbb{Z}/2$ algebra.

Remark 1.4.8. For the trivial group $\{e\}$ viewed as 0-dimensional Lie group we can identify $\mathfrak{N}_*^{\{e\}}$ with $\mathfrak{N}_*$, the (non-equivariant) non-oriented real cobordism ring.

Definition 1.4.9. A **singular G-manifold** over a pair of G-spaces (X, A) is a G-manifold M together with a G-map:

$$f : (M, \partial M) \to (X, A).$$

Definition 1.4.10. Two singular n-G-manifolds, (M_1, f_1) and (M_2, f_2), over (X, A) are **bordant** if there is an $(n+1)$-G-manifold W with two codimension-1 G-submanifolds $\partial_0 W$ and $\partial_1 W$ and a G-map

$$g : (W, \partial_1 W) \to (X, A)$$

such that ∂W is G-diffeomorphic to $\partial_0 W \cup \partial_1 W$, $\partial_0 W$ is G-diffeomorphic to $M_1 \amalg M_2$ with $g_{|\partial_0 W} = f_1 \amalg f_2$ and $\partial\partial_0 W = \partial_0 W \cap \partial_1 W = \partial\partial_1 W$.

Remark 1.4.11. "Bordant" gives an equivalence relation on singular manifolds over (X, A).

Definition 1.4.12. By $\mathfrak{N}_n^G(X, A)$ we denote the set of bordism classes of n-G-manifolds over a pair of spaces (X, A). For the pair $(X, \varnothing)$ we abbreviate

$$\mathfrak{N}_n^G(X) := \mathfrak{N}_n^G(X, \varnothing).$$

Remark 1.4.13. With respect to the disjoint union of manifolds and maps of representatives, the set $\mathfrak{N}_n^G(X, A)$ is an Abelian group, called the equivariant bordism group.

Definition 1.4.14. We set

$$\mathfrak{N}_*^G(X, A) := \bigoplus_{i=0}^{\infty} \mathfrak{N}_i^G(X, A).$$

Remark 1.4.15. With multiplication induced by taking product on representatives (a closed manifold with a singular manifold), $\mathfrak{N}_*^G(X,A)$ is a graded module over $\mathfrak{N}_*^G$.

Definition 1.4.16. For a G-map $\phi : (X_1, A_1) \to (X_2, A_2)$ we define

$$\phi_* : \mathfrak{N}_*^G(X_1, A_1) \to \mathfrak{N}_*^G(X_2, A_2)$$

by $\phi_*(M, f) := (M, \phi \circ f)$.

Remark 1.4.17. For a G-map $\phi : (X_1, A_1) \to (X_2, A_2)$, ϕ_* is a well-defined homomorphism of $\mathfrak{N}_*^G$-modules of degree 0.

Definition 1.4.18. We define a map

$$\partial : \mathfrak{N}_*^G(X,A) \to \mathfrak{N}_*^G(A)$$

by $\partial(M, f) := (\partial M, f_{|\partial M})$.

Remark 1.4.19. The map ∂ is a well-defined homomorphism of $\mathfrak{N}_*^G$-modules of degree -1.

Lemma 1.4.20. *Real equivariant bordism is a $\mathbb{Z}$-graded equivariant homology theory satisfying the dimension axiom $\mathfrak{N}_*^G(\mathrm{pt}) = \mathfrak{N}_*^G$.*

Proof. This is the unoriented equivariant analogue of [CF64, Theorem 5.1]. We will demonstrate exactness to give an idea how these kinds of proofs work.

For a pair of G-space (X, A) the sequence

$$\cdots \longrightarrow \mathfrak{N}_n^G(A) \xrightarrow{i_*} \mathfrak{N}_n^G(X) \xrightarrow{j_*} \mathfrak{N}_n^G(X,A) \xrightarrow{\partial} \mathfrak{N}_{n-1}^G(A) \longrightarrow \cdots$$

is exact, with i the inclusion $A \to X$ and j the inclusion $(X, \varnothing) \to (X, A)$.

At $\mathfrak{N}_n^G(X)$ exactness is immediate, since a bordism class of a singular G-manifold over (X, A) is in the kernel of j_* if and only if it is cobordant to a class represented by a singular G-manifold with image in A, that is, it is in the image of i_*.

At $\mathfrak{N}_n^G(X,A)$ we have $\partial j_* = 0$, since $\partial M = \varnothing$ for all $[M]$ in the image of j_*. For an element $[M, f]$ in the kernel of ∂ there is an n-G-manifold W with $\partial W = \partial M$ and all of W maps to A. Gluing together M and W along their boundary (and smoothing the corners) gives a singular G-manifold over $(X, \varnothing)$ that is mapped to M by i_*.

At $\mathfrak{N}_n^G(A)$ exactness is also immediate; an element $[M] \in \mathfrak{N}_n^G(A)$ is in the kernel of i_* and in the image of ∂ if and only if there is a singular $(n+1)$-G-manifold $(W, \partial W) \to (X, A)$ such that $\partial W = M$. □

Complex equivariant geometric (co)bordism

The complex analogue to $\mathfrak{N}_*^G$ is defined using the notion of stable almost complex G-manifold. For this concept we refer to Hanke [Han05, Definition 1]. In that paper a comparison is made between the different notions of stable almost complex G-structures, including the concept of normally almost complex G-structures and tangentially almost complex G-structures.

Definition 1.4.21. Let X be a G-space. By $\Omega_n^G(X)$ we denote the set of bordism classes of singular stable almost complex G-manifolds over X.

The complex analogue of Lemma 1.4.20 reads as follows.

Lemma 1.4.22. *Complex equivariant bordism $\Omega_*^G(-)$ is a $\mathbb{Z}$-graded equivariant homology theory satisfying the dimension axiom*

$$\Omega_*^G(\mathrm{pt}) = \Omega_*^G.$$

1.5 The Pontryagin–Thom construction

Another description of the coefficients for real equivariant homotopical bordism is easily derived from Lemma 1.3.14.

Remark 1.5.1. For a nonnegative integer k the coefficients of $\mathbb{Z}$-graded real equivariant homotopical bordism are given by

$$MO_k^G := \operatorname{colim}_{W \supset k}[S^W, TO^G(W-k)]_G$$

with maps as in Lemma 1.3.14.

We can restrict MO_*^G to a $\mathbb{Z}$-graded equivariant homology theory (compare Remark 1.2.39) and will use this notion and notation of MO_*^G from now on.

Definition 1.5.2. For every $k \in \mathbb{N}$ we construct a map:

$$PT : \mathfrak{N}_k^G \to MO_k^G$$

as follows. Given an element $[M]$ in $\mathfrak{N}_k^G$ represented by an k-G-manifold M, choose an embedding of M in a G-representation W. (The fact that this is possible is the Mostow-Palais theorem, see [Mos57] and [Pal57]. A proof is also given by Wasserman [Was69, §1]). Let ν be the normal bundle of the embedding. We can choose W such that ν is a G-bundle with fiber $W-k$, with total space $E(\nu)$ homeomorphic to a tubular neighborhood N of the image of M in W; compare [CF64, Chapter 3, Section 22]. (Remember that $W-k$ denotes the orthogonal complement of k in W.) We define a map

$$t : S^W \to T\nu$$

by sending N, viewed as a subset of S^W, to $E(\nu)$ viewed as a subset of $T\nu$ via the homeomorphism and send everything else, that is $S^W - N$, to the base point of $T\nu$. The normal bundle is classified by a map

$$f : E\nu \to EO(|W|-k).$$

This gives a map

$$Tf : T\nu \to TO^G(W-k)$$

and a homotopy class $[Tf \circ t] \in [S^W, TO^G(W-k)]_G$. The image of that class $[Tf \circ t]$ in the colimit

$$MO_k^G = \operatorname{colim}_{W \supset k}[S^W, TO^G(W-k)]_G$$

is defined to be $PT([M])$.

It can be shown that this gives a well-defined group homomorphism. The following generalization of the classical Pontryagin–Thom construction is due to tom Dieck, [tD71, §1]. Also compare [BH72, §3].

Theorem 1.5.3. *The above construction is well defined and a group homomorphism. It induces a ring homomorphism and a homomorphism of $\mathfrak{N}_*$-modules*

$$PT : \mathfrak{N}_*^G \to MO_*^G.$$

Furthermore the construction induces a natural transformation of $\mathbb{Z}$-graded equivariant homology theories

$$PT : \mathfrak{N}_*^G(-) \to MO_*^G(-).$$

Remark 1.5.4. We show in Section 1.8: Euler classes e_V of non-trivial representations V (which always exist if the group is non-trivial) are non-zero elements in $MO_{-|V|}^G$. We deduce that PT is not surjective for non-trivial G.

In the complex setting, a Pontryagin–Thom construction is obtained in a similar way. We get a ring homomorphism

$$PT : \Omega_*^G \to MU_*^G.$$

A detailed description can be found in [Han05, p. 681f].

1.6 A geometric description of MO_*^G

We can also consider stabilized geometric equivariant bordism.

Definition 1.6.1. Stabilized geometric bordism for a compact Lie group G is defined for every integer k as

$$\overline{MO}_k^G := \operatorname{colim}_V \mathfrak{N}_{k+|V|}^G(D(V), S(V))$$

where the colimit is taken over all indexing spaces and the maps in the colimit are taking the product with disc bundles. A bordism class represented by

$$f : (M, \partial M) \to (D(V), S(V))$$

maps to the bordism class of

$$\begin{aligned} f \times \mathrm{id}_{D(W)} : (M \times D(W), \partial(M \times D(W))) \\ \to (D(V) \times D(W), \partial(D(V) \times D(W))) \cong (D(V \oplus W), S(V \oplus W)). \end{aligned}$$

Similarly one defines $\overline{MO}_k^G(X, A)$ as a colimit over the groups

$$\mathfrak{N}_k^G((X, A) \times (D(V), S(V))).$$

Bröcker and Hook showed that the Pontryagin–Thom construction stabilizes to give the following natural transformation

$$\overline{\Phi} : \overline{MO}_k^G(X, A) \to MO_k^G(X, A).$$

Moreover they prove the following.

Theorem 1.6.2 (see [BH72, Theorem 4.1])**.** *For a compact Lie group G there is an isomorphism*

$$\overline{\Phi} : \overline{MO}_*^G(X, A) \to MO_*^G(X, A).$$

Over a point this gives the isomorphism

$$\operatorname{colim}_V \mathfrak{N}_{k+|V|}^G(D(V), S(V)) \cong MO_k^G.$$

As in the unstabilized Pontryagin–Thom construction we have a product structure on MO_*^G and the product on $\mathfrak{N}_*^G$ induces a product on $\overline{MO}_*^G$. Let V be a finite dimensional G-representation. Define $\chi(V)$ to be the element in $\overline{MO}_{-|V|}^G$ that is represented by the class of the map $[* \to D(V)]$ in

$$\mathfrak{N}_{-|V|+|V|}(D(V), S(V)) = \mathfrak{N}_0(D(V), S(V))$$

that sends $*$ to $0 \in D(V)$.

Remark 1.6.3. Chasing through the definitions of Bröcker and Hook's stabilized Pontryagin–Thom map we see that

$$\overline{\Phi}(\chi(V)) = e_V \in MO_{-|V|}^G.$$

Lemma 1.6.4 (compare [Cos96, p. 157])**.** *If V contains a trivial direct summand, then e_V is trivial in $MO_{-|V|}^G$.*

Proof. If V contains a trivial direct summand, then the map $* \to D(V)$ where $*$ maps to $0 \in D(V)$ is homotopic to a map $* \to S(V) \subset D(V)$ by going along a trivial direct summand of V. Hence $\chi(V)$ is zero in $\mathfrak{N}_{-|V|+|V|}$ and so is $e_V = \overline{\Phi}(\chi(V))$. □

The converse of this Lemma is also true and we will give a demonstration after introducing the notion of families of subgroups.

1.7 Families of subgroups

Remember that we assume all subgroups to be closed.

Definition 1.7.1. A **family of subgroups** $\mathcal{F}$ of G is a set of subgroups of G that is closed under conjugation (i.e. $H \in \mathcal{F}$ and $K \subset G$, $K \sim H$ implies $K \in \mathcal{F}$) and closed under taking subgroups (i.e. $H \in \mathcal{F}$ and $K \subset H$ implies $K \in \mathcal{F}$).

As examples we define special families of subgroups. Let G be a topological group.

Definition 1.7.2. The family of all subgroups in G is

$$\mathcal{A} := \{H \subset G \,|\, H \text{ closed subgroup in } G\}$$

and the family of all proper subgroups in G is

$$\mathcal{P} := \{H \subset G \,|\, H \neq G \text{ closed subgroup in } G\}.$$

Another example is the family of subgroups $\{\{e\}\}$ consisting only of the trivial subgroup $\{e\} \subset G$.

Definition 1.7.3. Let $\mathcal{F}$ be a family of subgroups. A G-space X is called **$\mathcal{F}$-numerable** if there is an open covering $U = \{U_j \,|\, j \in J\}$ of X by G-subspaces U_j such that

1. For all $j \in J$ there is a $G_j \in \mathcal{F}$ and a G-map
$$f_j : U_j \to G/G_j.$$
2. There is a (locally finite) partition of unity $(t_j)_{j \in J}$ subordinate to U by G-functions
$$t_j : X \to [0, 1].$$

Theorem 1.7.4 (see [tD72, Satz 1])**.** *Given a family of subgroups $\mathcal{F}$, the homotopy category of $\mathcal{F}$-numerable G-spaces has a terminal object $E\mathcal{F}$.*

This statement can be found in [MPC96, p. 45] and a proof can also be found in [tD87, Chapter I, Theorem 6.6].

Remark 1.7.5. The space $E\mathcal{F}$ is called *universal classifying space of G for the family $\mathcal{F}$* and is unique up to G-homotopy. It enjoys the following properties: $(E\mathcal{F})^H$ is (non-equivariantly) contractible for $H \in \mathcal{F}$ and it is empty for $H \notin \mathcal{F}$.

Remark 1.7.6. For $\mathcal{F} = \{\{e\}\}$, the family consisting only of the trivial subgroup, $E\{\{e\}\}$ can be identified with EG, the total space of the universal principal G-bundle.

Remark 1.7.7. For $\mathcal{F} = \mathcal{A}$, the family of all subgroups, we can take the space consisting only of one point, pt, as a model for $E\mathcal{A}$.

For more on these classifying space (especially a G-CW-structure) see [Lüc05].

Definition 1.7.8. Let $\mathcal{F}$ be a family of subgroups. An $\mathcal{F}$-space is G-space X such that all of the isotropy groups of X are in $\mathcal{F}$.

Definition 1.7.9. Let $(\mathcal{F}, \mathcal{F}')$ be a pair of families of subgroups with $\mathcal{F}' \subset \mathcal{F}$. Given an equivariant homology theory E_*^G we set

$$E_*^G[\mathcal{F}, \mathcal{F}'](X) := E_*^G(X \times E\mathcal{F}, X \times E\mathcal{F}')$$

to obtain the **equivariant homology theory associated to** $(\mathcal{F}, \mathcal{F}')$**.** If $\mathcal{F}'$ is empty we write $E_*^G[\mathcal{F}](X)$ instead of $E_*^G[\mathcal{F}, \varnothing](X)$.

Remark 1.7.10. By Remark 1.7.7, for the family $\mathcal{A}$ we have

$$E_*^G[\mathcal{A}](X) = E_*^G(X).$$

Remark 1.7.11. Let E_*^G be an equivariant homology theory and $\mathcal{F}' \subset \mathcal{F}$ families of subgroups. The long exact sequence of the pair $(X \times E\mathcal{F}, X \times E\mathcal{F}')$ gives a long exact sequence

$$\cdots \longrightarrow E_*^G[\mathcal{F}'](X) \longrightarrow E_*^G[\mathcal{F}](X) \longrightarrow E_*^G[\mathcal{F},\mathcal{F}'](X) \longrightarrow E_{*-1}^G[\mathcal{F}'](X) \longrightarrow \cdots .$$

There is also a notion of relative equivariant homology theory associated to $(\mathcal{F},\mathcal{F}')$; see below a more concrete example.

For real (or complex) geometric bordism there is a geometric interpretation of the homology theory associated to a pair $(\mathcal{F},\mathcal{F}')$ (compare [tD72, Satz 3]). The idea is that the families should contain all isotropy groups of manifolds representing elements in geometric bordism.

Definition 1.7.12. Let $\mathcal{F}' \subset \mathcal{F}$ be a pair of families of subgroups. An **$\mathcal{F}$-manifold** is a G-manifold M such that all isotropy groups of M are in $\mathcal{F}$. An **$(\mathcal{F},\mathcal{F}')$-manifold** is an $\mathcal{F}$-manifold such that ∂M is an $\mathcal{F}'$-manifold.

Now we can define a bordism between two $(\mathcal{F},\mathcal{F}')$-manifolds M_1 and M_2 to be an $\mathcal{F}$-manifold W with two codimension-1 submanifolds $\partial_0 W$ and $\partial_1 W$ such that

$$\begin{aligned}
&\partial W = \partial_0 W \cup \partial_1 W,\\
&\partial_0 W \cong M_1 \coprod M_2,\\
&\partial_1 W \text{ is an } \mathcal{F}'\text{-manifold and}\\
&\partial\partial_0 W = \partial_0 W \cap \partial_1 W = \partial\partial_1 W.
\end{aligned}$$

This can be identified with $\mathfrak{N}_*^G[\mathcal{F},\mathcal{F}']$ and considering singular $(\mathcal{F},\mathcal{F}')$-manifolds over X allows us to redefine $\mathfrak{N}_*^G[\mathcal{F},\mathcal{F}'](X)$ similarly.

For the definition of the relative geometric bordism groups with restricted isotropy

$$\mathfrak{N}_*^G[\mathcal{F},\mathcal{F}'](X,A)$$

we look at singular $\mathcal{F}$-manifold (M,f) over (X,A) together with two codimension 1 submanifolds $\partial_F M$ and $\partial_A M$ such that

$$\begin{aligned}
&\partial_F M \cup \partial_A M = \partial M,\\
&\partial_F M \text{ is an } \mathcal{F}\text{-manifold},\\
&f \text{ maps } \partial_A M \text{ to } A \subset X \text{ and}\\
&\partial\partial_F M = \partial_F M \cap \partial_A M = \partial\partial_A M.
\end{aligned}$$

A bordism between two such singular n-manifolds

$$(M_1, f_1, \partial_F M_1, \partial_A M_1) \text{ and } (M_2, f_2, \partial_F M_2, \partial_A M_2)$$

is an $\mathcal{F}$-manifold (W,g) of dimension $n+1$ and three codimension-1 submanifolds $\partial_0 W, \partial_1 W$ and $\partial_2 W$, such that

$$\begin{aligned}
&\partial W = \partial_0 W \cup \partial_1 W \cup \partial_2 W,\\
&\partial_0 W \cong M_1 \amalg M_2 \text{ and } g_{|\partial_0 W} \text{ restricts to } f_1 \amalg f_2,\\
&\partial_1 W \text{ is an } \mathcal{F}'\text{-manifold},\\
&g \text{ maps } \partial_2 W \text{ to } A \subset X \text{ and}\\
&\partial\partial_0 W = \partial(\partial_1 W \cup \partial_2 W) = \partial_0 W \cap (\partial_1 W \cup \partial_2 W).
\end{aligned}$$

Remark 1.7.13. The long exact sequence of the pair $(\mathcal{F},\mathcal{F}')$

$$\cdots \longrightarrow \mathfrak{N}_*^G[\mathcal{F}'](X) \xrightarrow{i_{\mathfrak{N}}} \mathfrak{N}_*^G[\mathcal{F}](X) \xrightarrow{j_{\mathfrak{N}}} \mathfrak{N}_*^G[\mathcal{F},\mathcal{F}'](X) \xrightarrow{\partial_{\mathfrak{N}}} \mathfrak{N}_{*-1}^G[\mathcal{F}'](X) \longrightarrow \cdots$$

of Remark 1.7.11 is called the *Conner–Floyd exact sequence* and has a geometric interpretation: ∂ is actually taking boundaries of singular $(\mathcal{F},\mathcal{F}')$-manifolds.

Remark 1.7.14. For real homotopical equivariant bordism $MO^G_*(-)$ the corresponding long exact sequence

$$\cdots \longrightarrow MO^G_*[\mathcal{F}'](X) \xrightarrow{i_{MO}} MO^G_*[\mathcal{F}](X) \xrightarrow{j_{MO}} MO^G_*[\mathcal{F},\mathcal{F}'](X) \xrightarrow{\partial_{MO}} MO^G_{*-1}[\mathcal{F}'](X) \longrightarrow \cdots$$

is called the *tom Dieck exact sequence.*

Remark 1.7.15. The natural transformation given by the Pontryagin–Thom construction (Theorem 1.5.3) can be applied between the Conner–Floyd exact sequence and tom Dieck exact sequence to obtain a commutative diagram with exact rows:

$$\begin{tikzcd}
\cdots \arrow[r] \arrow[d] & \mathfrak{N}^G_*[\mathcal{F}'](X) \arrow[r] \arrow[d, "PT[\mathcal{F}']"] & \mathfrak{N}^G_*[\mathcal{F}](X) \arrow[r] \arrow[d, "PT[\mathcal{F}]"] & \mathfrak{N}^G_*[\mathcal{F},\mathcal{F}'](X) \arrow[r] \arrow[d, "PT[\mathcal{F},\mathcal{F}']"] & \mathfrak{N}^G_{*-1}[\mathcal{F}'](X) \arrow[r] \arrow[d, "PT[\mathcal{F}']"] & \cdots \arrow[d] \\
\cdots \arrow[r] & MO^G_*[\mathcal{F}'](X) \arrow[r] & MO^G_*[\mathcal{F}](X) \arrow[r] & MO^G_*[\mathcal{F},\mathcal{F}'](X) \arrow[r] & MO^G_{*-1}[\mathcal{F}'](X) \arrow[r] & \cdots .
\end{tikzcd}$$

The complex case is analogous. References are [MPC96, p. 158, p. 339] and [CF64, Section 5, p. 13].

Considering the pair $(\mathcal{A},\mathcal{P})$ for geometric bordism has a useful interpretation.

Proposition 1.7.16 (compare [CF66, Lemma (5.2) and Theorem (7.3)])**.** *The group $\mathfrak{N}^G_n[\mathcal{A},\mathcal{P}]$ is isomorphic to the group of bordism classes of G-vector bundles over trivial base spaces, with the property that each fibre does not contain a trivial direct summand.*

sketch proof. In a first step we notice that a manifold N, that represents an element $[N] \in \mathfrak{N}^G_n[\mathcal{A},\mathcal{P}]$ is bordant to every tubular neighborhood of its fixed set $M := N^G$, which lies in the interior of N, since there are no fixed points on the boundary. This can be seen by a straigthening-the-angle argument and then giving the bordism explicitly. The normal bundle of the embedding of M into N is G-homeomorphic to a such a small tubular neighborhood. This normal bundle is a G-bundle with the property that each fiber is a G-representation that does not contain a trivial G-representation as a direct summand. □

1.8 Euler classes of non-trivial representations

As an example of the use of families we show that for non-trivial G, the Pontryagin–Thom map is not surjective by showing the existence of non-trivial elements of negative degree and so prove the converse of Lemma 1.6.4. This is precisely [Cos96, Lemma 3.1]). Costenoble does not give a complete proof but an indication how one could proceed. Let G be a compact Lie group and V an indexing space.

Proposition 1.8.1 (see [Cos96, Lemma 3.1])**.** *If V contains no trivial direct summand, then the Euler class e_V is a non-trivial element in $MO^G_{-|V|}$.*

Proof. We use the geometric description of $MO^G_{-|V|}$ (see Section 1.6) and show that $\chi(V)$ is non-zero in $\overline{MO}^G_{-|V|} = \overline{MO}^G_{-|V|}[\mathcal{A}]$ (compare Remark 1.7.10). (These $\chi(V)$ are identified with the Euler classes e_V, see Remark 1.6.3.) We consider

$$j : \overline{MO}^G_*[\mathcal{A}] \to \overline{MO}^G_*[\mathcal{A},\mathcal{P}]$$

and show that $j(\chi(V))$ is invertible. In $\overline{MO}^G_{-|V|}[\mathcal{A}]$ the element $\chi(V)$ is represented by the 0-dimensional singular manifold $* \to D(V)$ over $D(V)$ sending $*$ to $0 \in D(V)$. We have

$$\overline{MO}^G_{-|V|}[\mathcal{A},\mathcal{P}] = \operatorname{colim}_W \mathfrak{N}^G_{-|V|+|W|}[\mathcal{A},\mathcal{P}](D(W),S(W))$$

and $j(\chi(V)) \in \overline{MO}^G_{-|V|}[\mathcal{A},\mathcal{P}]$ is also represented by $* \to D(V)$. We can give an inverse to $j(\chi(V))$ and since $\overline{MO}^G_*[\mathcal{A},\mathcal{P}]$ is non-zero it follows that $j(\chi(V))$ is non-zero. The inverse is an element in

$$\overline{MO}^G_{|V|}[\mathcal{A},\mathcal{P}] = \operatorname{colim}_W \mathfrak{N}_{|V|+|W|}[\mathcal{A},\mathcal{P}](D(W),S(W))$$

represented by the map $D(V) \to *$. Notice that here we use that V contains no trivial direct summand; $S(V)$ has no fixed points and $[D(V) \to *]$ really lies in $\mathfrak{N}^G_{|V|}[\mathcal{A},\mathcal{P}]$. The product of $j(\chi(V))$ and its proposed inverse is defined by taking the product of the representing singular manifolds. Hence it is represented by the map $D(V) \to D(V)$ in $\mathfrak{N}^G_{|V|}[\mathcal{A},\mathcal{P}](D(V),S(V))$ that maps all of $D(V)$ to $0 \in D(V)$. We show that this is bordant to the element represented by the identity $D(V) \to D(V)$, which represents the unit in

$$\overline{MO}^G_* \supset \overline{MO}^G_0[\mathcal{A},\mathcal{P}] = \operatorname{colim}_W \mathfrak{N}^G_{|V|}[\mathcal{A},\mathcal{P}](D(V),S(V)).$$

The bordism is given by the singular manifold $W := D(V) \times I$ over $(D(V),S(V))$ with the following map:

$$\begin{aligned} W = D(V) \times I &\to D(V) \\ (x,t) &\mapsto \begin{cases} 2tx & t \le \frac{1}{2}, \\ x & t \ge \frac{1}{2}, \end{cases} \end{aligned}$$

$$\begin{aligned} \partial_0 W &:= D(V) \times \{0,1\} \\ \partial_1 W &:= S(V) \times [0,\tfrac{1}{2}] \\ \partial_2 W &:= S(V) \times [\tfrac{1}{2},1]. \end{aligned}$$

We check that this really gives the desired bordism. Clearly

$$\begin{aligned} \partial W = \partial(D(V) \times I) &= (D(V) \times \partial I) \cup (\partial D(V) \times I) \\ &= (D(V) \times \{0,1\}) \cup (S(V) \times I) \\ &= (D(V) \times \{0,1\}) \cup (S(V) \times [0,\tfrac{1}{2}]) \cup (S(V) \times [\tfrac{1}{2},1]) = \partial_0 W \cup \partial_1 W \cup \partial_2 W. \end{aligned}$$

Also $\partial_1 W = S(V) \times [0,\frac{1}{2}]$ has no fixed point, because V does not contain a trivial direct summand and

$$\partial_2 W = S(V) \times [\tfrac{1}{2},1]$$

is mapped to $S(V) \subset D(V)$. One easily checks that this gives the desired bordism between the map $D(V) \to D(V)$ with image 0 (on $D(V) \times \{0\} \subset \partial_0 W$) and the identity $D(V) \to D(V)$ (on $D(V) \times \{1\} \subset \partial_0 W$). So $j(\chi(V))$ really is invertible, hence $\chi(V)$ and e_V are indeed non-zero. □

Lemma 1.6.4 and Proposition 1.8.1 give the following.

Proposition 1.8.2. *For a compact Lie group G and an indexing space V, the Euler class e_V is trivial if and only if V contains a trivial direct summand.*

This justifies Remark 1.5.4 about the non-surjectivity of the Pontryagin–Thom map for non-trivial G.

2 Known results

In this chapter we give an overview of known results without giving proofs. Mainly we state results about the Pontryagin–Thom map in its non-equivariant, real equivariant and complex equivariant form.

2.1 Non-equivariant bordism

Our description of the Pontryagin–Thom construction (Section 1.5) specializes to the non-equivariant theory if we take G to be the trivial group. The classical result is by Thom.

Theorem 2.1.1 (see [Tho54, Théorème IV.12])**.** *The real (non-equivariant) Pontryagin–Thom map*

$$PT : \mathfrak{N}_* \to MO_*$$

is an isomorphism and

$$\mathfrak{N}_* \cong MO_* \cong \mathbb{Z}/2[u_k \,|\, k \neq 2^t - 1, t > 0]$$

with generators u_k of degree k for $k \geq 2$.

Other proofs of this theorem can be found in [Liu62, Theorem 2], [Sto68, Chapter 2], [May99, Chapter 25, Section 2] and in [Swi75, Theorem 12.30]. In his book, Stong also gives a proof that the *complex* (non-equivariant) Pontryagin–Thom map is an isomorphism [Sto68, Chapter 2]. The key step in the proofs are transversality arguments. For non-trivial G, the Pontryagin–Thom map fails to be an isomorphism due to the lack of transversality when considering equivariant maps. As an example of this failure consider $G = \mathbb{Z}/2$ and let N be $\mathbb{R}$ with the non-trivial $\mathbb{Z}/2$-action and $M = \{*\}$ be the 0-dimensional manifold consisting of a point with trivial $\mathbb{Z}/2$-action. The map $f : M \to N$ that sends $*$ to $0 \in N$ is equivariant and not $\mathbb{Z}/2$-homotopic to a map transverse to the inclusion of 0 into N, since it is the only equivariant map from M to N and itself not transverse to the inclusion of 0 into N.

2.2 Real equivariant bordism

Spectral sequences for equivariant homology theories

We follow Costenoble [Cos96, Section 3]. For an equivariant homology theory E_*^G and a sequence $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{A}$ of families of subgroups of G with $\bigcup_{i=0}^{\infty} \mathcal{F}_i = \mathcal{A}$ we obtain an (unraveled) exact couple (compare [Boa99])

$$\begin{tikzcd}
\cdots \arrow[r, dotted] & E_*^G[\mathcal{F}_{p-2}] \arrow[rr, "i"] \arrow[dl, dotted] & & E_*^G[\mathcal{F}_{p-1}] \arrow[rr, "i"] \arrow[dl, "j"] & & E_*^G[\mathcal{F}_p] \arrow[rr, "i"] \arrow[dl, "j"] & & E_*^G[\mathcal{F}_{p+1}] \arrow[dl, "j"] & \cdots \\
{} & & E_*^G[\mathcal{F}_{p-1}, \mathcal{F}_{p-2}] \arrow[ul, "k"] & & E_*^G[\mathcal{F}_p, \mathcal{F}_{p-1}] \arrow[ul, "k"] & & E_*^G[\mathcal{F}_{p+1}, \mathcal{F}_p] \arrow[ul, "k"] & & {} \arrow[ul, dotted]
\end{tikzcd}$$

coming from the exact sequence of Remark1.7.11; the maps i and j have degree 0 and the degree of k is -1. This gives a spectral sequence with $E^1_{p,q} := E^G_q[\mathcal{F}_p, \mathcal{F}_{p-1}]$.

For complex equivariant bordism this spectral sequence is discussed for example in [Row78, Section 2].

For real equivariant bordism we get $E^1_{p,q} := \mathfrak{N}^G_q[\mathcal{F}_p, \mathcal{F}_{p-1}]$. Costenoble mentions the following result in [Cos96, Section 3, p. 160].

Proposition 2.2.1. *The above spectral sequence converges to* $\mathfrak{N}^G_*[\mathcal{A}] = \mathfrak{N}^G_*$.

More details can be found by Wasserman [Was66, Theorem] and tom Dieck [tD72, §3]. This helps to reduce the calculation of $\mathfrak{N}^G_*[\mathcal{A}]$ to a non-equivariant problem; the groups $\mathfrak{N}^G_*[\mathcal{F}_p, \mathcal{F}_{p-1}]$ can be expressed as sum of non-equivariant bordism groups if the families $\mathcal{F}_p$ and $\mathcal{F}_{p-1}$ are adjacent, i.e. they only differ by a single conjugacy class of a subgroup of G. A simple example of that is the calculation of $\mathfrak{N}^G_*[\{e\}, \varnothing]$, the bordism group of free closed G-manifolds. From the free G-manifold we pass to the manifold M/G and by considering the classifying space BG of G we obtain the isomorphism $\mathfrak{N}^G_n[\{e\}] \cong \mathfrak{N}_{n-\dim G}(BG)$. Another reduction to a non-equivariant bordism problem can be found by considering the pair of families $\mathcal{P} \subset \mathcal{A}$, which is one of the key ingredients of Chapter 3.

$G = \mathbb{Z}/2$

For the case $G = \mathbb{Z}/2$ a sequence of families of subgroups is given by

$$\varnothing \subset \{\{e\}\} \subset \{\{e\}, \mathbb{Z}/2\}.$$

With this sequence of families the spectral sequence of Section 2.2 collapses after the first step and we only need to consider short exact sequences. This is how Sinha proceeds in [Sin02]. As in Section 3 a comparison of the tom Dieck exact sequence and the Conner–Floyd exact sequence is used. Sinha gives generators for $MO^{\mathbb{Z}/2}_*$ over $\mathfrak{N}_*$ and relations. Also the following description of $\mathfrak{N}^{\mathbb{Z}/2}_*$ is given.

Theorem 2.2.2 (see [Sin02, Theorem 2.7])**.** *The ring* $\mathfrak{N}^{\mathbb{Z}/2}_*$ *is the subring of* $MO^{\mathbb{Z}/2}_*$ *generated by classes* $PT(g_{i,n})$ *for certain geometric elements* $g_{i,n} \in \mathfrak{N}^{\mathbb{Z}/2}_*$.

Sinha also describes the quotient $\mathfrak{N}^{\mathbb{Z}/2}_*$-module

$$MO^{\mathbb{Z}/2}_*/\mathfrak{N}^{\mathbb{Z}/2}_*,$$

which can be interpreted as transversality obstructions.

The equivariant Pontryagin–Thom map

One major ingredient for the proof of Theorem 3.5.2 is the observation, that the equivariant Pontryagin–Thom map is injective for certain groups. This has been shown by tom Dieck.

Theorem 2.2.3 (see [tD71, Theorem 2])**.** *For* $G = \mathbb{Z}/2 \times \cdots \times \mathbb{Z}/2$, *the Pontryagin–Thom map*

$$PT : \mathfrak{N}^G_*(X, A) \to MO^G_*(X, A)$$

is a monomorphism.

The proof uses induction on the number of factors of $\mathbb{Z}/2 \times \cdots \times \mathbb{Z}/2$. The key step is a construction of a map similar to our map $\phi_{\mathfrak{N}}$ in Section 3.1 together with the localization techniques of Section 3.2. We use Theorem 2.2.3 in the following two cases.

Corollary 2.2.4. *For* $G = \mathbb{Z}/2 \times \cdots \times \mathbb{Z}/2$, *the Pontryagin–Thom map*

$$PT : \mathfrak{N}^G_* \to MO^G_*$$

is a monomorphism.

Corollary 2.2.5. *Let* $\mathcal{P}$ *denote the family of proper subgroups of* $G = \mathbb{Z}/2 \times \cdots \times \mathbb{Z}/2$. *Then the Pontryagin–Thom map*

$$PT : \mathfrak{N}^G_*[\mathcal{P}] \to MO^G_*[\mathcal{P}]$$

is a monomorphism.

2.3 Complex equivariant bordism

Let G be a compact Lie group. In complex equivariant cobordism there are a couple of interesting results. The Pontryagin–Thom map

$$PT : \Omega_*^G \to MU_*^G$$

fails to be an isomorphism for non-trivial G due to a lack of surjectivity. On the other hand there is the following Theorem by tom Dieck.

Theorem 2.3.1 (see [tD70, Proposition 1.3])**.** *For a free G-space X, the equivariant complex Pontryagin–Thom map*

$$PT : \Omega_*^G(X) \to MU_*^G(X)$$

is an isomorphism.

Injectivity of the Pontryagin–Thom map has been much studied.

Proposition 2.3.2. *The complex equivariant Pontryagin–Thom map*

$$PT : \Omega_*^G \to MU_*^G$$

is a monomorphism for certain types of groups G described below.

For cyclic groups of prime order this was shown by tom Dieck [tD70, Theorem 5.1(c)].

More generally for G a compact Abelian Lie group it was shown by Comezaña [Com96, p. 342], that the Pontryagin–Thom map is a *split* monomorphism of MU_*-modules. This and the following proposition was claimed by Löffler [Löf73] and proved in his (unpublished) thesis.

Proposition 2.3.3. *For G a compact Abelian Lie group, MU_*^G is a free MU_*-module concentrated in even degrees.*

Proofs of this proposition can be found in [Lan72, Theorem 1], [Oss72, Theorem 1], [Löf74, Satz 5.8] and also in [Com96, p. 342].

Toral G

For toral $G = S^1 \times \cdots \times S^1$ we mention Hanke's result [Han05, Theorem 1] in the beginning of the next Chapter and state it already here:

Theorem 2.3.4. *There is a pullback square*

$$\begin{array}{ccc} \Omega_*^G & \longrightarrow & MU_*[e_V^{-1}, Y_{d,V}] \\ \downarrow & & \downarrow \\ MU_*^G & \longrightarrow & MU_*[e_V, e_V^{-1}, Y_{d,V}] \end{array}$$

with all maps injective, for certain elements e_V, e_V^{-1} and $Y_{d,V}$, where V runs through a set J of representations containing exactly one representative of every isomorphism class of irreducible non-trivial representations and $1 + |V| \leq d$.

3 Real equivariant bordism for $G = \mathbb{Z}/2 \times \cdots \times \mathbb{Z}/2$

Hanke shows in [Han05, Theorem 1] for the group $T = S^1 \times \cdots \times S^1$ the existence of a pullback square

$$\begin{tikzcd}
\Omega_*^T \arrow[r] \arrow[d] & MU_*[e_V^{-1}, Y_{d,V}] \arrow[d] \\
MU_*^T \arrow[r] & MU_*[e_V, e_V^{-1}, Y_{d,V}]
\end{tikzcd}$$

with injective maps, for certain elements e_V, e_V^{-1} and $Y_{d,V}$, where V runs through a set J of representations containing exactly one representative of every isomorphism class of irreducible non-trivial representations and $1 + |V| \leq d$. The goal of this chapter is to parallel his paper to obtain a similar result for *real* equivariant bordism.

3.1 The fixed set of the Thom space

For the rest of this chapter we take G to be $\mathbb{Z}/2 \times \cdots \times \mathbb{Z}/2 = (\mathbb{Z}/2)^k$.

Remark 3.1.1. Although only this case is important in the following, the statements in Section 3.1 are still true for compact Abelian Lie groups and the proofs are very similar. However, for infinite groups, the definition of B which follows has to be slightly altered; see [Han05, p. 683]. In Sections 3.2-3.5 the assumption

$$G = \mathbb{Z}/2 \times \cdots \times \mathbb{Z}/2$$

is imperative as counterexamples demonstrate; see Chapter 4.

For $G = \mathbb{Z}/2 \times \cdots \times \mathbb{Z}/2 = (\mathbb{Z}/2)^k$ a complete set J of representations containing exactly one representative of every isomorphism class of non-trivial irreducible representations consists of $2^k - 1$ elements; $J = \{V_i\}_{1 \leq i \leq 2^k - 1}$. We choose a base point $1 \in BO$ (see Remark 1.3.18) and set

$$B := BO^{\times |J|} = BO^{\times (2^k - 1)}.$$

The H-space structure on BO induces an H-space structure on B.

The following fact can be found (without proof) in [Sin01, Proposition 4.6].

Remark 3.1.2. The space B classifies G-bundles without fixed points in every fiber over a base space X with trivial G-action. Such a bundle $E \to X$ can be written as

$$\bigoplus_{V \in J} E_V \otimes_{\mathbb{R}} V \to X$$

for real vector bundles E_V. This bundle is classified by a map $X \to B$, where the map in the V-th component classifies E_V.

The statement is a classical result by Segal [Seg68, Proposition 2], and for G a finite group Oliver [Oli96, Appendix] also gives a proof.

We construct a map $\phi_{MO} : MO^G_* \to MO_*[e_V, e_V^{-1}, X_{d,V}]$ and need a few identifications to proceed. Consider the fixed point set of the Thom space of the n-dimensional equivariant universal bundle $(T(\xi^G_n))^G$.

Proposition 3.1.3 (compare [Sin01, Proposition 4.7])**.** *We have the following homotopy equivalence:*

$$(T(\xi^G_n))^G \simeq \bigvee_{\substack{W \in RO^+(G) \\ |W|=n}} T(\xi_{|W^G|}) \wedge \left(\prod_{V \in J} BO(\nu_V(W)) \right)_+ ,$$

where $RO^+(G)$ is a set of G-representations, containing one from every isomorphism class. It can be seen as a subset of $RO(G)$. The number of times V appears as a direct summand of W is denoted by $\nu_V(W)$.

Proof. The space $(BO^G(n))^G$ classifies n-dimensional G-vector bundles E over a base space X with trivial G-action. Such an E decomposes according to Remark 3.1.2 as follows:

$$E \cong \bigoplus_{V \in J \cup \{1\}} E_V \otimes_{\mathbb{R}} V.$$

We conclude that a path component F of E over X with fiber W is classified by a map to

$$\prod_{V \in J \cup \{1\}} BO(\nu_V(W)),$$

where the map to the factor $BO(\nu_V(W))$ is a classifying map of E_V. Hence the path components of $(BO^G(n))^G$ are those spaces and

$$(BO^G(n))^G \simeq \coprod_{W \in RO^+(G)} \left(\prod_{V \in J \cup \{1\}} BO(\nu_V(W)) \right).$$

The universal bundle over each component

$$\prod_{V \in J \cup \{1\}} BO(\nu_V(W)) = BO(|W^G|) \times \prod_{V \in J} BO(\nu_V(W))$$

is the product $\xi_{|W^G|} \times \xi$, where $\xi_{|W^G|}$ is the $|W^G|$-dimensional universal bundle and ξ is a G-vector bundle with non-trivial action in each fiber, so $E(\xi)^G$ is the zero section $\prod_{V \in J} BO(\nu_V(W))$. Passing to Thom spaces gives

$$(T(\xi^G_n))^G \simeq \bigvee_{\substack{W \in RO^+(G) \\ |W|=n}} T(\xi_{|W^G|}) \wedge \left(\prod_{V \in J} BO(\nu_V(W)) \right)_+ .$$

$\square$

Proposition 3.1.4 (compare [Sin01, Theorem 4.9])**.** *There is an equivalence of ring-spectra*

$$\Phi^G MO^G \simeq I_{RO(G)} \wedge MO \wedge B_+$$

with

$$I_{RO(G)} := \bigvee_{\substack{W \in RO(G) \\ |W|=0}} S^{|W^G|}.$$

Here $I_{RO(G)}$ carries the structure of a ring spectrum induced by the isomorphism

$$S^{|W^G|} \wedge S^{|V^G|} \to S^{|(W \oplus V)^G|}$$

for elements $W, V \in RO(G)$, $|W| = |V| = 0$.

Proof. As in the definition of geometric fixed point spectrum (see Section 1.2, Definition 1.2.56) we choose a sequence of representations $(K_n)_{n \geq 0}$. Proposition 3.1.3 allows us to make the identification

$$(\phi^G TO^G)_n = T(\xi^G_{|K_n|})^G \simeq \bigvee_{\substack{W \in RO^+(G)\\ |W|=|K_n|=n}} T(\xi_{|W^G|}) \wedge \left(\prod_{V \in J} BO(\nu_V(W)) \right)_+ .$$

The structure maps send the wedge summand of a representation $W \in RO^+(G)$ to the summand of a representation W' with $W - K_n = W' - K_{n+1}$ in $RO(G)$. The virtual dimension of $W - K_n = W' - K_{n+1}$ is zero. This leads to a splitting of $\Phi^G TO^G$ as a wedge sum; each summand corresponding to an element in $RO(G)$ with virtual dimension zero. For one wedge summand indexed by W the structure map is the structure map of the prespectrum TO^G on the first factor smashed with inclusions of the $BO(\nu_V(W))$s on the second factor.

Passing to spectra we obtain a copy of $MO \wedge (\prod_{V \in J} BO)_+$ suspended by $S^{|V|}$, where V corresponds the wedge summand indexed by $S^{|W-K_n|}$.

The ring structure on $\phi^G MO^G$ is induced by the ring structure on MO^G and checking how the identification of Proposition 3.1.3 behaves on smash products, we see that our equivalence is compatible with the ring structures. □

Proposition 3.1.5 (compare [Han05, p. 684])**.** *There is an isomorphism of graded rings*

$$(I_{RO(G)} \wedge MO \wedge B_+)_* \xrightarrow{\cong} MO_*(B) \otimes AO_*(G) .$$

Proof. In the proof of Proposition 3.1.4 we describe $I_{RO(G)} \wedge MO \wedge B_+$ as suspended copies of $MO \wedge B_+$. For such a copy indexed by an element $W - U \in RO(G)$ of virtual dimension zero with $W = W^G \oplus (W^G)^\perp$ and $U = U^G \oplus (U^G)^\perp$ we identify $(S^{(W-U)^G} \wedge MO \wedge B_+)_*$ with

$$MO_*(B) \otimes (e_{(W^G)^\perp} \cdot e^{-1}_{(U^G)^\perp}) \subset MO_*(B) \otimes AO_*(G).$$

This induces the desired isomorphism.

□

Proposition 3.1.6 (compare [Sin01, Theorem 4.10])**.** *There is an isomorphism of graded rings*

$$MO_*(B) \otimes AO_*(G) \cong MO_*[e_V, e_V^{-1}, Y_{d,V}],$$

where V runs through a set J of representations containing exactly one representative of every non-trivial irreducible representations of G and d runs through the integers such that

$$1 + |V| \leq d.$$

Proof. Since the set J is finite we have the isomorphism of the Künneth formula

$$MO_*(\prod_{V \in J} BO) \cong \bigotimes_{MO_*}^{V \in J} MO_*(BO).$$

Conner and Floyd calculated $MO_*(-)$ using the Atiyah–Hirzebruch spectral sequence, see [CF64, Theorem 8.3, Theorem 17.1]. For $MO_*(BO)$ we obtain

$$\begin{aligned} MO_*(BO) &= MO_*(\operatorname{colim}_n BO(n)) \\ &\cong \operatorname{colim}_n MO_*(BO(n)) \\ &\cong \operatorname{colim}_n MO_* \otimes_{\mathbb{Z}/2} H_*(BO(n), \mathbb{Z}/2) \\ &\cong \operatorname{colim}_n MO_* \otimes_{\mathbb{Z}/2} \mathbb{Z}/2[X_1, \dots, X_n] \\ &\cong \operatorname{colim}_n MO_*[X_1, \dots, X_n] \\ &\cong MO_*[X_i]_{1 \le i \le \infty}, \end{aligned}$$

where each generator X_i has degree i. The generators X_i can be represented by a map

$$\mathbb{R}P^i \to BO(1) \to BO$$

classifying the tautological line bundle $E_i \to \mathbb{R}P^i$, (compare [Koc96, Proposition 2.3.7 and 2.4.3].)

This is all we need to know to get the isomorphism:

$$\begin{aligned} MO_*(B) \otimes AO_*(G) &\cong MO_*(\prod_{V \in J} BO) \otimes_{\mathbb{Z}} \mathbb{Z}[e_V, e_V^{-1}]_{V \in J} \\ &\cong \bigotimes_{MO_*}^{V \in J} MO_*(BO) \otimes_{\mathbb{Z}} \mathbb{Z}[e_V, e_V^{-1}]_{V \in J} \\ &\cong \bigotimes_{MO_*}^{V \in J} MO_*[X_{i,V}]_{1 \le i \le \infty} \otimes_{\mathbb{R}} \mathbb{Z}[e_V, e_V^{-1}]_{V \in J} \\ &\cong MO_*[e_V, e_V^{-1}, X_{i,V}]_{V \in J, 1 \le i \le \infty} \\ &\cong MO_*[e_V, e_V^{-1}, X_{d-|V|,V} \cdot e_V^{-1}]_{V \in J, 1+|V| \le d}. \end{aligned} \tag{$\star$}$$

Definition 3.1.7. For a G-representation $V \in J$ and $1 + |V| \le d \le \infty$ we set

$$Y_{d,V} := X_{d-|V|,V} \cdot e_V^{-1}.$$

We identify $Y_{d,V}$ with the image of

$$X_{d-|V|} \otimes e_V^{-1} \in MO_{d-|V|}(BO) \otimes AO_{|V|}(G)$$

under the inclusion of BO as V-th factor in B viewed, via the isomorphism ($\star$), as an element in $MO_*[e_V, e_V^{-1}, X_{i,V}]$. With this definition we get the desired isomorphism

$$MO_*(B) \otimes AO_*(G) \cong MO_*[e_V, e_V^{-1}, Y_{d,V}]_{V \in J, 1+|V| \le d}.$$

□

Notice that $Y_{d,V}$ is defined in such a way, that its dimension is d. From now on we shorten

$$MO_*[e_V, e_V^{-1}, Y_{d,V}]_{V \in J, 1+|V| \le d}$$

in our notation to

$$MO_*[e_V, e_V^{-1}, Y_{d,V}].$$

Remark 3.1.8. In the introduction of Hanke's paper [Han05, p. 678], analogous classes are introduced: "[...] certain classes $Y_{V,d}$, $2 \le d < \infty$ [...]". Later in the paper it is clarified that $|V| + 1 \le d$.

Definition 3.1.9. We combine the results of Proposition 3.1.4 to 3.1.6 to define a map

$$\phi_{MO} : MO_*^G \to MO_*[e_V, e_V^{-1}, Y_{d,V}]$$

as follows:

$$MO_*^G \xrightarrow[\text{Remark 1.2.57}]{} \phi^G MO_*^G$$

$$\xrightarrow[\text{Proposition 3.1.4}]{\cong} (I_{RO(G)} \wedge MO \wedge B_+)_*$$

$$\xrightarrow[\text{Proposition 3.1.5}]{\cong} MO_*(B) \otimes AO_*(G) \xrightarrow[\text{Proposition 3.1.6}]{\cong} MO_*[e_V, e_V^{-1}, Y_{d,V}]$$

where the first map is the map restricting to fixed points. Not including the last isomorphism we get a map

$$\tilde{\phi}_{MO} : MO_*^G \to MO_*(B) \otimes AO_*(G).$$

We check that our notation e_V for two different notions of e_V makes sense. Euler classes e_V (see Definition 1.3.22) are mapped to indeterminates e_V (see 1.2.50), that now appear in $MO_*[e_V, e_V^{-1}, Y_{d,V}]$ and come from the identification

$$AO_*(G) \cong \mathbb{Z}[e_V, e_V^{-1}].$$

Proposition 3.1.10 (compare [Han05, p. 685])**.** *For an irreducible non-trivial representation V and the corresponding Euler class $e_V \in MO_*^G$ we have*

$$\phi_{MO}(e_V) = e_V \in MO_*[e_V, e_V^{-1}, Y_{d,V}].$$

Proof. The Euler class is represented by a map $S^0 \to TO^G(|V|)$. Reducing to fixed sets gives a map

$$S^0 \to (TO^G(|V|))^G = \bigvee_{\substack{W \in R^+(G)\\ |W|=|V|}} T(\xi_{|W^G|}) \wedge \left(\prod_{V \in J} BO(\nu_V(W))\right).$$

Going through the definition of ϕ_{MO}, under the isomorphism

$$\phi^G MO_*^G \cong (I_{RO(G)} \wedge MO \wedge B_+)_*,$$

the Euler class is sent to $1 \in MO_*(B)$, in the copy of $MO_*(B)$ suspended by $S^{V-|V|}$. Under the next isomorphism

$$(I_{RO(G)} \wedge MO \wedge B_+)_* \cong MO_*(B) \otimes AO_*(G)$$

it is mapped to $1 \otimes e_{V^{G\perp}} \cdot e_{|V|^{G\perp}}^{-1} = 1 \otimes e_V \cdot e_0^{-1} = 1 \otimes e_V$, which is mapped to e_V under the last identification

$$MO_*(B) \otimes AO_*(G) \cong MO_*[e_V, e_V^{-1}, Y_{d,V}].$$

□

Next we want to construct a map

$$\phi_{\mathfrak{N}} : \mathfrak{N}_*^G \to MO_*[e_V, e_V^{-1}, Y_{d,V}].$$

Let M^n be a manifold representing an element $[M] \in \mathfrak{N}_*^G$ and let $F \subset M^G$ be a connected component of the fixed set of M. Then F is embedded in M. The normal bundle ν_F^M of F in M is a real G-vector bundle of dimension d, with non-trivial G-action in each fiber. This bundle decomposes as follows:

$$\nu_F^M = \bigoplus_{k=1}^{j} E_k \otimes_{\mathbb{R}} V_k$$

for real vector bundles E_j and irreducible G-representations V_j. Define

$$b_F := \overline{b}_F \otimes (e_{V_1}^{-|E_1|} \cdot \dots \cdot e_{V_j}^{-|E_j|}) \in MO_{n-d}(B) \otimes AO_d(G),$$

where $\overline{b}_F \in MO_{n-d}(B)$ is represented by a map $F \to B$ with V_k-th component the classifying map for E_k. Altogether we get a map

$$\tilde{\phi}_{\mathfrak{N}} : \mathfrak{N}_*^G \to MO_*(B) \otimes AO_*(G)$$

by setting

$$\tilde{\phi}_{\mathfrak{N}}([M]) := \sum_{F \subset M^G} b_F \in (M(B) \otimes AO(G))_n.$$

Compare tom Dieck's description of the map in [tD71, Section 5]. Composing with the isomorphism of Proposition 3.1.6 we get a map

$$\phi_{\mathfrak{N}} : \mathfrak{N}_*^G \to MO_*[e_V, e_V^{-1}, Y_{d,V}].$$

For the next theorem we construct a map $MO_*(B) \to MO_*(B)$.

Definition 3.1.11 (compare [tD70, p. 354])**.** The inverse of the H-space B structure gives a map

$$-^{-1} : B \to B.$$

This induces a map

$$\nu : MO_*(B) \to MO_*(B),$$

which has order 2. Together with the isomorphism of Proposition 3.1.6, ν induces a map

$$\iota : MO_*[e_V, e_V^{-1}, Y_{d,V}] \to MO_*[e_V, e_V^{-1}, Y_{d,V}],$$

such that the following diagram commutes:

$$\begin{tikzcd} MO_*(B) \otimes AO(G) \arrow[r, "\cong"] \arrow[d, "\nu\otimes\mathrm{id}"] & MO_*[e_V, e_V^{-1}, Y_{d,V}] \arrow[d, "\iota"] \\ MO_*(B) \otimes AO(G) \arrow[r, "\cong"] & MO_*[e_V, e_V^{-1}, Y_{d,V}]. \end{tikzcd}$$

Remark 3.1.12. Notice for the complex analogue to our Proposition 3.1.10, namely the statement

$$\iota \circ \phi_{MU} \circ \Psi([P(\mathbb{C}^d \oplus V)]) = Y_{V,d} + e_{V^*}^{-d},$$

in the notation used there [Han05, p. 685], the analogous map ι is used. However, since

$$(\nu \otimes \mathrm{id})(1 \otimes e_V) = \nu(1) \otimes e_V = 1 \otimes e_V,$$

we have

$$\iota \circ \phi_{MO}(e_V) = e_V = \phi_{MO}(e_V).$$

(In general $\iota \circ \phi_{MO} \neq \phi_{MO}$.)

Theorem 3.1.13 (compare [tD70, Proposition 4.1])**.** *The following diagram commutes:*

$$\begin{tikzcd} \mathfrak{N}_*^G \arrow[r, "\tilde{\phi}_{\mathfrak{N}}"] \arrow[d, "PT"] & MO_*(B) \otimes AO_*(G) \\ MO_*^G \arrow[r, "\tilde{\phi}_{MO}"] & MO_*(B) \otimes AO_*(G). \arrow[u, "\nu\otimes\mathrm{id}"] \end{tikzcd}$$

Proof. Let M^n be a manifold representing an element $[M] \in \mathfrak{N}_n^G$. We embed M in a real G-representation $U = U^G \oplus \bigoplus_{j\in J} V_j^{k_j}$. A map $f : S^U \to T(\xi_m^G)$ then represents $PT([M])$, where m is the codimension-$|U| - n$ of M in U. This map, when restricted to M, classifies the normal bundle ν_M^U of M in U. Now we chase through the definition of $\tilde{\phi}_{MO}$. Restricting f to fixed points gives a map

$$f^G : S^{U^G} \to \bigvee_{\substack{W\in RO^+(G)\\ W=W^G\oplus\bigoplus_{j\in J} V_j^{m_j}\\ |W|=m}} T(\xi_{|W^G|}) \wedge \left(\prod_{j\in J} BO(m_j)\right)_+ ,$$

which is transverse to

$$\coprod_{\substack{W\in RO^+(G)\\ W=W^G\oplus\bigoplus_{j\in J} V_j^{m_j}\\ |W|=m}} BO(|W^G|) \times \prod_{j\in J} BO(m_j).$$

(See [CW92, Section 2] and for more on transversality.) The pre-image under f^G is the fixed point set $F = M^G$. Restricting f^G to F gives a classifying map for $\nu_{M|F}^U$. If we denote by F_W the parts of F over which $\nu_{M|F}^U$ has fiber W we get a decomposition $F = \coprod F_W$ and f^G decomposes into a sum of maps

$$f_W^G : \nu_{M|F_W}^U \to BO(|W^G|) \times \prod_{j\in J} BO(m_j).$$

Since $\nu_{M|F_W}^U$ is a vector bundle over a trivial base we have a decomposition

$$\nu_{M|F_W}^U \cong D_{\underline{1}}^W \oplus \bigoplus_{j\in J} D_j^W \otimes_{\mathbb{R}} V_j$$

and if $d_j^W : F_W \to BO(|D_j^W|)$ is a classifying map for D_j^W we get a map $d^W : F_W \to B$. Then $\tilde{\phi}_{MO} \circ PT([M])$ is

$$\sum_W [d^W] \otimes \left(\prod_{j\in J} e_{V_j}^{m_j} \cdot e_{V_j}^{-k_j}\right) \in MO_*(B) \otimes AO_*(G).$$

Recalling the definition of $\tilde{\phi}_{\mathfrak{N}}$ we have a decomposition of the normal bundle of the embedding $F \to M$ for every fiber W:

$$\nu_{F\ |F_W}^M = \bigoplus_{j\in J} E_j^W \otimes_{\mathbb{R}} V_j^W$$

and if $e_j^W : F_W \to BO(|E_j|)$ is a classifying map for E_j^W we get a map $e^W : F_W \to B$. Then $\tilde{\phi}_{\mathfrak{N}}([M])$ is

$$\sum_W [e^W] \otimes \left(\prod_{j\in J} e_{V_j^W}^{-|E_j^W|}\right) \in MO_*(B) \otimes AO_*(G).$$

We have the following embeddings:

$$\begin{tikzcd} & M \arrow[dr] & \\ F \arrow[ur] \arrow[dr] & & U . \\ & U^G \arrow[ur] & \end{tikzcd}$$

Let $\nu_F^{U^G}$ denote the normal bundle of F in U^G, which is trivial and let $\nu_{U^G}^U$ denote the normal bundle of U^G in U, which is $U^{G\perp}$. Considering the normal bundle of F in U gives an isomorphism of bundles:

$$\nu_F^{U^G} \oplus \nu_{U^G}^U \cong \nu_F^U \cong \nu_F^M \oplus \nu_M^U.$$

Restricting to F_W gives:

$$\nu_F^{U^G}{}_{|F_W} \oplus \nu_{U^G|F_W}^{U} \cong \bigoplus E_j^W \otimes_{\mathbb{R}} V_j^W \oplus \left(D_{\underline{1}}^W \oplus \bigoplus_{j\in J} D_j^W \otimes_{\mathbb{R}} V_j \right).$$

It follows that $D_{\underline{1}}^W \cong \nu_F^{U^G}{}_{|F_W}$ and the bundle inverse to E_j^W is equivalent to the bundle D_j^W; hence νe_j^W is homotopic to d_j^W and $[\nu e^W] = [d^W]$. Considering the fiber, the isomorphism of bundles above yields:

$$W^G \oplus \bigoplus_{j\in J} V_j^{k_j} = \bigoplus_{j\in J} V_j^{|E_j|} \oplus W^G \oplus \bigoplus_{j\in J} V_j{}^{m_j}.$$

Counting dimensions gives: $k_j = |E_j| + m_j$ and hence $e_{V_j}^{-|E_j|} = e_{V_J}^{m_j} \cdot e_{V_j}^{-k_j}$. Together we have as desired isomorphism

$$\sum_W [d^W] \otimes \left(\prod_{j\in J} e_{V_j}^{m_j} \cdot e_{V_j}^{-k_j} \right) = \sum_W [\nu e^W] \otimes \left(\prod e_{V_j^W}^{-|E_j^W|} \right).$$

□

3.2 Localization

The goal of this section is to give an alternative description of ϕ_{MO}. The key step is a localization result by tom Dieck.

Proposition 3.2.1 (see [tD71, Theorem 1(b)]). *Let S be the set of Euler classes of non-trivial irreducible representations in MO_*^G. Then the localization map into the ring of quotients*

$$\lambda : MO_*^G \to S^{-1} MO_*^G$$

is injective.

Compare the complex version of this Proposition [Sin01, Corollary 5.2].

The map

$$\tilde{\phi}_{MO} : MO_*^G \to MO_*(B) \otimes AO_*(B)$$

sends all elements in S to units. This can be seen by considering

$$\phi_{MO} : MO_*^G \to MO_*[e_V, e_V^{-1}, Y_{d,V}].$$

Clearly the image $\phi_{MO}(e_V)$ of an Euler class e_V (compare section 1.8) is a unit in $MO_*[e_V, e_V^{-1}, Y_{d,V}]$. Hence the universal property of localization gives a unique map

$$\tilde{\Phi}_{MO} : S^{-1} MO_*^G \to MO_*(B) \otimes AO_*(B)$$

such that $\tilde{\Phi}_{MO} \circ \lambda = \tilde{\phi}_{MO}$. We cite the following result without a proof.

Proposition 3.2.2 (see [tD71, p. 217]). *The map*

$$\tilde{\Phi}_{MO} : S^{-1} MO_*^G \to MO_*(B) \otimes AO_*(B)$$

is an isomorphism.

Complex versions of the propositions are [tD70, Theorem 3.1] and [Sin01, Corollary 4.15]. For $G = \mathbb{Z}/2$ the corresponding statement is [Sin02, Corollary 3.19]. Fitting it all together and composing with the isomorphism of Proposition 3.1.6 gives the following commutative diagram.

$$\begin{tikzcd}
MO_*^G \arrow[r, "\lambda"] \arrow[dr, "\tilde{\phi}_{MO}"] \arrow[ddr, bend right, "\phi_{MO}"'] & S^{-1}MO_*^G \arrow[d, "\tilde{\Phi}_{MO}"] \arrow[dd, bend left=60, "\Phi_{MO}"] \\
& MO_*(B) \otimes AO_*(G) \arrow[d, "\text{Proposition 3.1.6}"', "\cong"] \\
& MO_*[e_V, e_V^{-1}, Y_{d,V}]
\end{tikzcd}$$

Corollary 3.2.3. *We have*

$$\Phi_{MO} \circ \lambda = \phi_{MO}$$

and ϕ_{MO} is a monomorphism.

3.3 The geometric image

Proposition 3.3.1 (compare [Han05, Proposition 3])**.** *The image of $\phi_{\mathfrak{N}}$ lies in the subalgebra*

$$MO_*[e_V^{-1}, Y_{d,V}] \subset MO_*[e_V, e_V^{-1}, Y_{d,V}].$$

Proof. Let M^n be a manifold representing an element $[M] \in \mathfrak{N}_*^G$ and let $F \subset M^G$ be a connected component of the fixed set of M. As in the definition of $\phi_{\mathfrak{N}}$, the normal bundle of F in M decomposes as follows:

$$\nu_F^M = \bigoplus_{k=1}^{j} E_k \otimes_{\mathbb{R}} V_k.$$

We proceed to show that

$$b_F = \overline{b}_F \otimes V_1^{-|E_1|} \cdot \dots \cdot V_j^{-|E_j|} \in MO_{n-k}(B) \otimes AO_k(G) \cong MO_*[e_V, e_V^{-1}, Y_{d,V}]$$

already lies in $MO[e_V^{-1}, Y_{d,V}]$ by inspection of $\overline{b}_F$. This element in $MO_{n-k}(B)$ is represented by a map

$$F \to BO(|E_1|) \times \dots \times BO(|E_j|)$$

so $\overline{b}_F$ lies in

$$\begin{aligned}
MO_*(BO(|E_1|) \times \dots \times BO(|E_j|)) &\cong \bigotimes_{MO_*}^{1 \le k \le q} MO_*(BO(|E_j|)) \\
&\cong \bigotimes_{MO_*}^{1 \le k \le q} MO_*[X_1, \dots, X_{|E_k|}] \\
&\subset \bigotimes_{MO_*}^{1 \le k \le q} MO_*[X_{d,V_k}]_{d>0}.
\end{aligned}$$

In fact every element in $MO_*(BO(|E_j|))$ can be written as a sum of monomials with at most $|E_j|$ factors X_{d,V_j}. (Compare the classical calculations in [CF64, Theorem 8.3] and [Koc96, Propositions 2.4.3 and 2.3.7].) By definition of the $Y_{d,V}$'s we have $X_{d,V} = Y_{d+|V|,V} \cdot e_V$ and this asserts that e_{V_j} appears at most $|E_j|$ times as factor in $\overline{b}_F$ and hence appears in nonnegative degree (i.e. with non-positive exponent) in $\overline{b}_F$. Together b_F lies in $MO_*[e_V^{-1}, Y_{d,V}]$ and $\phi_{\mathfrak{N}}([M])$ is just a sum of elements b_F, so it also lies in $MO_*[e_V^{-1}, Y_{d,V}] \subset MO_*[e_V, e_V^{-1}, Y_{d,V}]$. $\square$

3.4 Families of subgroups and isomorphisms

Proposition 3.4.1 (compare [Han05, Proposition 4])**.** *There is an isomorphism*

$$\kappa_{\mathfrak{N}} : \mathfrak{N}_*^G[\mathcal{A}, \mathcal{P}] \to MO_*[e_V^{-1}, Y_{d,V}]$$

such that the following diagram commutes:

$$\begin{tikzcd}
 & & MO_*[e_V^{-1}, Y_{d,V}] \\
\mathfrak{N}_*^G[\mathcal{A}] \arrow[rru, "\phi_{\mathfrak{N}}"] \arrow[r, "j_{\mathfrak{N}}"'] & \mathfrak{N}_*^G[\mathcal{A}, \mathcal{P}] \arrow[ru, "\kappa_{\mathfrak{N}}"'] &
\end{tikzcd}.$$

The map $j_{\mathfrak{N}}$ comes from the Conner–Floyd exact sequence (see Remark 1.7.13).

Proof. First we define a map $\tilde{\kappa}_{\mathfrak{N}} : \mathfrak{N}_*^G[\mathcal{A}, \mathcal{P}] \to MO_*[e_V, e_V^{-1}, Y_{d,V}]$. An element $[N] \in \mathfrak{N}_n^G[\mathcal{A}, \mathcal{P}]$ can be viewed as as vector bundle $E \to M$ of dimension m over a connected component M (of dimension $n - m$) of its fixed set N^G of dimension $n - m$ (see Proposition 1.7.16). This is a vector bundle over a trivial base. From here we proceed as in the definition of $\phi_{\mathfrak{N}}$. This vector bundle has a decomposition:

$$E = \bigoplus_{k=1}^{j} E_k \otimes_{\mathbb{R}} V_k.$$

Using the classifying maps for E_j we get a map

$$M \to BO(|E_1|) \times \cdots \times BO(|E_j|) \to B,$$

which gives an element $\bar{b}_M \in MO_{n-m}(B)$ and finally an element

$$\kappa_{\mathfrak{N}}([N]) := b_M := \bar{b}_M \otimes e_{V_1}^{-|E_1|} \cdot \cdots \cdot e_{V_j}^{-|E_j|} \in MO_{n-m} \otimes AO_m(G).$$

Composing with the isomorphism of Proposition 3.1.6 we get a map

$$\kappa_{\mathfrak{N}} : \mathfrak{N}_*^G[\mathcal{A}, \mathcal{P}] \to MO_*[e_V, e_V^{-1}, Y_{d,V}].$$

The image of $\kappa_{\mathfrak{N}}$ already lies in $MO_*[e_V^{-1}, Y_{d,V}]$. This is proved exactly as was Proposition 3.3.1. To see that $\kappa_{\mathfrak{N}}$ is an isomorphism we give an inverse

$$\kappa_{\mathfrak{N}}^{-1} : MO_*[e_V^{-1}, Y_{d,V}] \to \mathfrak{N}_*^G[\mathcal{A}, \mathcal{P}].$$

The element e_V^{-1} is sent to the class of the disc bundle of V viewed as a bundle over a point. Since V does not contain the trivial representation its unit disc bundle $D(V)$ has boundary $S(V)$ without fixed points. Then $\kappa_{\mathfrak{N}}$ sends this bundle back to e_V^{-1}, since we have the decomposition $\mathbb{R} \otimes_{\mathbb{R}} V \to *$ and the class of the map $* \to BO(1)$ classifying $\mathbb{R}$ gives $1 \in MO_*(B)$, so

$$\kappa_{\mathfrak{N}}([V \to *]) = 1 \otimes e_V^{-|\mathbb{R}|} = e_V^{-1}.$$

On $Y_{d,V}$ the inverse $\kappa_{\mathfrak{N}}^{-1}$ is constructed as follows: Let $E_{d-|V|}$ denote the line bundle representing the generator $X_{d-|V|}$ (compare the proof of Proposition 3.1.6). Then $\kappa_{\mathfrak{N}}^{-1}(Y_{d,V})$ is defined to be the class of the disc bundle of $E_{d-|V|} \otimes V$. As above we get

$$\kappa_{\mathfrak{N}}([E_{d-|V|} \otimes V]) = X_{d-|V|} \otimes e_V^{|E_{d-|V|}|} = X_{d-|V|} \otimes e_V^{-1} = Y_{d,V}.$$

Now $\kappa_{\mathfrak{N}}^{-1}$ is defined by requiring it to be a homomorphism of $\mathfrak{N}_*$-modules and a ring homomorphism. Clearly $\kappa_{\mathfrak{N}}^{-1}$ is a right and a left inverse of $\kappa_{\mathfrak{N}}$.

The commutativity follows immediately from the construction of $\kappa_{\mathfrak{N}}$.

□

Proposition 3.4.2 (compare [Han05, Proposition 4]). *There is an isomorphism*

$$\kappa_{MO} : MO_*^G[\mathcal{A},\mathcal{P}] \to MO_*[e_V, e_V^{-1}, Y_{d,V}]$$

such that the following diagram commutes:

$$\begin{tikzcd} & & MO_*[e_V, e_V^{-1}, Y_{d,V}] \\ MO_*^G[\mathcal{A}] \arrow[rru, "\phi_{MO}"] \arrow[r, "j_{MO}"'] & MO_*^G[\mathcal{A},\mathcal{P}] \arrow[ru, "\kappa_{MO}"'] & \end{tikzcd}.$$

The map j_{MO} comes from the tom Dieck exact sequence (see Remark 1.7.14).

Proof. By definition $MO_*^G[\mathcal{A},\mathcal{P}]$ is $MO_*^G(E\mathcal{A},E\mathcal{P})$. Let $\overline{\Sigma}E\mathcal{P}$ be the unreduced suspension of $E\mathcal{P}$:

$$\overline{\Sigma}E\mathcal{P} := ([0,1]\times E\mathcal{P})/{\scriptstyle (0,x)\sim(0,y) \atop (1,x)\sim(1,y)}.$$

We identify $E\mathcal{P}$ with $\frac{1}{2}\times E\mathcal{P}\subset \overline{\Sigma}E\mathcal{P}$ and denote the upper cone by $C^+E\mathcal{P} := [\frac{1}{3},1]\times E\mathcal{P}\subset\overline{\Sigma}E\mathcal{P}$ and the lower cone by $C^-E\mathcal{P} := [0,\frac{2}{3}]\times E\mathcal{P}$. Then $(E\mathcal{A},E\mathcal{P})\simeq(C^-E\mathcal{P},E\mathcal{P})$ and the inclusion

$$(C^-E\mathcal{P},E\mathcal{P})\to(\overline{\Sigma}E\mathcal{P},C^+E\mathcal{P})$$

gives an isomorphism via excision:

$$MO_*^G(E\mathcal{A},E\mathcal{P})\cong MO_*^G(\overline{\Sigma}E\mathcal{P},C^+E\mathcal{P}).$$

To calculate $MO_*^G(\overline{\Sigma}E\mathcal{P},C^+E\mathcal{P}) = MO_*^G(\overline{\Sigma}E\mathcal{P})$ we apply Lemma 4.2 of [Sin01]:

Lemma 3.4.3 (see [Sin01, Lemma 4.2]). *Let Z be a G-complex such that $Z^G\simeq S^0$ and Z^H is contractible for any proper subgroup $H\subsetneq G$. For a finite G-complex X the restriction map*

$$(\mathrm{Map}(X,Y\wedge Z))^G\to\mathrm{Map}(X^G,(Y\wedge Z)^G)=\mathrm{Map}(X^G,Y^G)$$

is a homotopy equivalence.

For the G-complex $\overline{\Sigma}E\mathcal{P}$ (compare Section 1.7) and any proper subgroup $H\subsetneq G$, the space $(\overline{\Sigma}E\mathcal{P})^H$ is contractible by the construction of $E\mathcal{P}$ and furthermore

$$(\overline{\Sigma}E\mathcal{P})^G\simeq S^0.$$

Since S^W is a finite G-complex, we obtain

$$\begin{aligned} MO_n^G(\overline{\Sigma}E\mathcal{P}) &\cong \mathrm{colim}_W[S^W, T(\xi^G_{|W|+n})\wedge\overline{\Sigma}E\mathcal{P}]^G \\ &\cong \mathrm{colim}_W[(S^W)^G,(T(\xi^G_{|W|+n}))^G] \\ &\cong \Phi^G MO_n^G. \end{aligned}$$

Combining this with the isomorphism $\Phi^G MO_n^G\cong MO_*[e_V,e_V^{-1},Y_{d,V}]$ of Propositions 3.1.4 to 3.1.6, we get the desired isomorphism κ_{MO}. To show commutativity of the following diagram, we look at the definitions of ϕ_{MO} and κ_{MO}:

$$\begin{tikzcd} MO_n^G(E\mathcal{A}) = MO_*^G[\mathcal{A}] = MO_*^G \arrow[r, "\text{Remark 1.2.57}"] \arrow[d, "j"] & \Phi^G MO_n^G \\ MO_*^G(E\mathcal{A},E\mathcal{P}) \arrow[r, "\cong"] & MO_*^G(\overline{\Sigma}E\mathcal{P}). \arrow[u, "\cong"'] \end{tikzcd}$$

Here the upper horizontal map is the first map in the definition of ϕ_{MO}; it is restriction to fixed sets (see Definition 3.1.9). The vertical map on the left hand side j comes from the tom Dieck exact sequence (compare Remark 1.7.14). Let f represent an element in $MO_n^G = MO_n^G(E\mathcal{A})$,

$$f : S^W \to TO_n(\xi^G_{n+|W|}) \wedge E\mathcal{A}.$$

Restricting to fixed sets gives an element represented by

$$f^G : (S^W)^G \to (TO_n(\xi^G_{n+|W|}))^G.$$

On the other hand we see that

$$\begin{aligned} j(f) \in [S^W, T(\xi^G_{n+|W|}) \wedge E\mathcal{A}/E\mathcal{P}]^G &= [(S^W)^G, (T(\xi^G_{n+|W|}) \wedge (\overline{\Sigma} E\mathcal{P}))^G] \\ &= [(S^W)^G, (T(\xi^G_{n+|W|}))^G] \end{aligned}$$

gives the same element and the diagram commutes. Combining this with isomorphisms of Propositions 3.1.4 to 3.1.6 we get $\kappa_{MO} \circ j = \phi_{MO}$.

□

We can combine the diagram of Proposition 3.4.2 with the isomorphism ι.

Corollary 3.4.4. *The following diagram commutes:*

$$\begin{tikzcd} & MO_*[e_V, e_V^{-1}, Y_{d,V}] \\ MO_*^G[\mathcal{A}] \arrow[ur, "\iota\circ\phi_{MO}"] \arrow[r, "j_{MO}"'] & MO_*^G[\mathcal{A},\mathcal{P}] \arrow[u, "\iota\circ\kappa_{MO}"'] \end{tikzcd}.$$

3.5 Conclusion

Proposition 3.5.1 (compare [Han05, Proposition 4])**.** *The following diagram commutes:*

$$\begin{tikzcd} \mathfrak{N}_*^G[\mathcal{A},\mathcal{P}] \arrow[r, "\kappa_{\mathfrak{N}}"] \arrow[d, "PT{[\mathcal{A},\mathcal{P}]}"] & MO_*[e_V^{-1}, Y_{d,V}] \arrow[d, "i"] \\ MO_*^G[\mathcal{A},\mathcal{P}] \arrow[r, "\iota\circ\kappa_{MO}"] & MO_*[e_V, e_V^{-1}, Y_{d,V}] \end{tikzcd}$$

Proof. This is essentially the same as the proof of Theorem 3.1.13. Notice that ι corresponds to ν there (see Definition 3.1.11). Given an element $[N]$ in $\mathfrak{N}_n^G[\mathcal{A},\mathcal{P}]$ we construct the element $i \circ \kappa_{\mathfrak{N}}([N])$ using the same notation as in the definition of $\kappa_{\mathfrak{N}}$. Then

$$i \circ \kappa_{\mathfrak{N}}([N]) = \bar{b}_M \otimes e_{V_1}^{-|E_1|} \cdot \cdots \cdot e_{V_j}^{-|E_j|} \in MO_{n-k} \otimes AO_k(G).$$

On the other hand we choose an embedding $N \to U$ into a G-representation

$$U = U^G \oplus \bigoplus_{i\in I} V_i$$

and $PT[\mathcal{A},\mathcal{P}]([N])$ is then represented by a map

$$S^U \to T(\xi^G_{|U|-n}) \wedge E\mathcal{P}$$

classifying the normal bundle of the embedding. Considering the map

$$S^{U^G} \to T(\xi^G_{|U|-k})^G$$

viewed as an element in $\phi^G MO^G$ we obtain the element $\kappa_{MO} \circ PT[\mathcal{A},\mathcal{P}]([N])$. As before we examine the normal bundles of the embeddings

$$\begin{tikzcd}
& N \arrow[dr] & \\
M \arrow[ur] \arrow[dr] & & U \\
& U^G \arrow[ur] &
\end{tikzcd}$$

and get the desired conclusion

$$\iota \circ \kappa_{MO} \circ PT[\mathcal{A},\mathcal{P}]([N]) = i \circ \kappa_{\mathfrak{N}}([N]).$$

□

Theorem 3.5.2 (compare [Han05, Theorem 1])**.** *The following diagram commutes and is a pull-back with all maps injective:*

$$\begin{tikzcd}
\mathfrak{N}^G_* \arrow[r, "\phi_{\mathfrak{N}}"] \arrow[d, "PT"] & MO_*[e_V^{-1}, Y_{d,V}] \arrow[d, "i"] \\
MO^G_* \arrow[r, "\iota\circ\phi_{MO}"] & MO_*[e_V, e_V^{-1}, Y_{d,V}]
\end{tikzcd}$$

Proof. Putting together the exact sequences of Remarks 1.7.11 and 1.7.15 and the commutative diagrams of Propositions 3.4.1, 3.5.1 and Corollary 3.4.4, gives the following commutative diagram with exact horizontal rows:

$$\begin{tikzcd}
& & MO_n[e_V^{-1}, Y_{d,V}] \arrow[dd, "i"] & \\
\mathfrak{N}^G_n[\mathcal{A}] \arrow[urr, "\phi_{\mathfrak{N}}"] \arrow[r, "j_{\mathfrak{N}}"'] \arrow[dd, "PT[\mathcal{A}]"] & \mathfrak{N}^G_n[\mathcal{A},\mathcal{P}] \arrow[ur, "\kappa_{\mathfrak{N}}"', "\cong"] \arrow[rr, "\partial_{\mathfrak{N}}"] \arrow[dd, "PT[\mathcal{A},\mathcal{P}]"] & & \mathfrak{N}^G_{n-1}[\mathcal{P}] \arrow[dd, "PT[\mathcal{P}]"] \\
& & MO_n[e_V, e_V^{-1}, Y_{d,V}] & \\
MO^G_n[\mathcal{A}] \arrow[urr, "\iota\circ\phi_{MO}"] \arrow[r, "j_{MO}"'] & MO^G_n[\mathcal{A},\mathcal{P}] \arrow[ur, "\iota\circ\kappa_{MO}"', "\cong"] \arrow[rr, "\partial_{MO}"] & & MO^G_{n-1}[\mathcal{P}].
\end{tikzcd}$$

Using the isomorphisms $\kappa_{\mathfrak{N}}$ and $\iota \circ \kappa_{MO}$ to substitute

$$\begin{tikzcd}
\mathfrak{N}^G_n[\mathcal{A},\mathcal{P}] \arrow[d, "PT[\mathcal{A},\mathcal{P}]"] \\
MO^G_n[\mathcal{A},\mathcal{P}]
\end{tikzcd}$$

in the middle by the inclusion

$$\begin{tikzcd} MO_n[e_V^{-1}, Y_{d,V}] \arrow[d, "i"] \\ MO_n[e_V, e_V^{-1}, Y_{d,V}], \end{tikzcd}$$

gives the following commutative diagram with short exact sequences as rows:

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{N}_n^G \arrow[r, "\phi_{\mathfrak{N}}"] \arrow[d, "PT"] & MO_n[e_V^{-1}, Y_{d,V}] \arrow[r, "\partial_{\mathfrak{N}} \circ \kappa_{\mathfrak{N}}^{-1}"] \arrow[d, "i"] & \mathfrak{N}_{n-1}^G[\mathcal{P}] \arrow[r] \arrow[d, "PT[\mathcal{P}]"] & 0 \\ 0 \arrow[r] & MO_n^G \arrow[r, "\iota \circ \phi_{MO}"] & MO_n^G[e_V, e_V^{-1}, Y_{d,V}] \arrow[r, "\partial_{MO} \circ (\iota \circ \kappa_{MO})^{-1}"] & MO_{n-1}^G[\mathcal{P}] \arrow[r] & 0. \end{tikzcd}$$

The Pontryagin–Thom maps $PT = PT[\mathcal{A}]$ and $PT[\mathcal{P}]$ are injective by Section 2.2 and so is the inclusion in the middle. From that and the injectivity of $\iota \circ \phi_{MO}$ (see Section 3.2) the injectivity of $\phi_{\mathfrak{N}}$ follows. (The injectivity of $\phi_{\mathfrak{N}}$ can also be deduced from the injectivity of $j_{\mathfrak{N}}$; see Proposition 4.1.1). To prove the pullback property it suffices to show that an element $x \in \operatorname{im} i \cap \operatorname{im} \iota \circ \phi_{MO}$ comes from an element in $\mathfrak{N}_n^G$. Let $y \in MO_n[e_V^{-1}, Y_{d,V}]$ and let $z \in MO_n^G$ be an element such that $i(y) = x = \iota \circ \phi_{MO}(z)$. Then

$$\begin{aligned} PT[\mathcal{P}] \circ (\partial_{\mathfrak{N}} \circ \kappa_{\mathfrak{N}}^{-1})(y) &= (\partial_{MO} \circ (\iota \circ \kappa_{MO})^{-1}) \circ i(y) \\ &= (\partial_{MO} \circ \kappa_{MO}^{-1} \circ \iota^{-1}) \circ (\iota \circ \phi_{MO})(z) \\ &= (\partial_{MO} \circ \kappa_{MO}^{-1} \circ \phi_{MO})(z) = (\partial_{MO} \circ j_{MO})(z) = 0 \end{aligned}$$

by exactness of the lower row and hence $(\partial_{\mathfrak{N}} \circ \kappa_{\mathfrak{N}}^{-1})(y) = 0$ since $PT[\mathcal{P}]$ is injective. By exactness of the upper row y is in the image of $\phi_{\mathfrak{N}}$ and we get an element $m \in \mathfrak{N}_n^G$ with $\phi_{\mathfrak{N}}(m) = y$ and $PT(m) = z$ as desired. □

We identify $MO_*[e_V^{-1}, Y_{d,V}]$ as a subring of $MO_*[e_V, e_V^{-1}, Y_{d,V}]$ via i.

Corollary 3.5.3 (compare [Han05, Corollary 1]). *The following isomorphism of MO_*-algebras describes geometric equivariant bordism for $G = \mathbb{Z}/2 \times \cdots \times \mathbb{Z}/2$:*

$$\mathfrak{N}_*^G \cong \iota \circ \phi_{MO}(MO_*^G) \cap MO_*[e_V^{-1}, Y_{d,V}].$$

3.6 Comparison with Sinha's results for $G = \mathbb{Z}/2$

The description of $MO_*^{\mathbb{Z}/2}$ in [Sin02, Theorem 2.4] is more explicit than ours in Theorem 3.5.2. In both cases $MO_*^{\mathbb{Z}/2}$ is identified with a subring of
$MO_*[e_\sigma, e_\sigma^{-1}, Y_{d,\sigma}]$. Here σ denotes the non-trivial one-dimensional real representation of $\mathbb{Z}/2$.

Also the description of $\mathfrak{N}_*^{\mathbb{Z}/2}$ in Theorem 2.7 of [Sin02] is more explicit than ours, but the generators given there can be derived from the pullback property of our Theorem 3.5.2 and Theorem 2.4 of [Sin02].

4 Counterexamples

Theorem 3.5.2 fails to be true if G is not of the form $\mathbb{Z}/2 \times \cdots \times \mathbb{Z}/2$. For the complex case Hanke shows that his Theorem 2.3.4 [Han05, Theorem 1] does not hold if G is not of the form $S^1 \times \cdots \times S^1$. He gives counterexamples for $G = \mathbb{Z}/n \times \mathbb{Z}/n$ and $G = \mathbb{Z}/n^2$ [Han05, Section 4]. In the real case the situation is similar. There are different ways Theorem 3.5.2 can fail for G not of the form $\mathbb{Z}/2 \times \cdots \times \mathbb{Z}/2$. The claim for $G = (\mathbb{Z}/2)^k$ for some k is that the diagram

$$\begin{tikzcd}
\mathfrak{N}_*^G \arrow[r, "\phi_{\mathfrak{N}}"] \arrow[d, "PT"] & MO_*[e_V^{-1}, Y_{d,V}] \arrow[d, "i"] \\
MO_*^G \arrow[r, "\iota\circ\phi_{MO}"] & MO_*[e_V, e_V^{-1}, Y_{d,V}]
\end{tikzcd}$$

is a pull-back and all maps are injective. In Section 4.1 we show that $\phi_{\mathfrak{N}}$ also fails to be injective for G not of the form $\mathbb{Z}/2 \times \cdots \times \mathbb{Z}/2$. In this sense, our Theorem 3.5.2 is the best possible, since it does not hold for other types of groups. For $G = \mathbb{Z}/4$ we give an explicit counterexample. In Section 4.2 we show that $\iota \circ \phi_{MO}$ fails to be injective for $G = \mathbb{Z}/4$. Clearly i is injective for all groups, but PT might not be injective for certain groups; compare Question 5.2.6. If this was the case, the description would also cease to be a pull-back. We do not know if there is a group G, such that the diagram is not a pull-back; even in a case where PT remains injective; see Question 5.2.4.

4.1 Geometric failure

Proposition 4.1.1. *The homomorphism*

$$j_{\mathfrak{N}} : \mathfrak{N}_*^G[\mathcal{A}] \to \mathfrak{N}_*^G[\mathcal{A}, \mathcal{P}]$$

from the Conner–Floyd exact sequence (see Remark 1.7.11) is a monomorphism if and only $G = (\mathbb{Z}/2)^k$ for some k.

Proof. Stong proves in [Sto70b, Proposition 14.2, p. 75], that

$$\iota_{\mathfrak{N}} : \mathfrak{N}_*^G[\mathcal{P}] \to \mathfrak{N}_*^G[\mathcal{A}]$$

is trivial if and only if $G = (\mathbb{Z}/2)^k$ for some k. (One direction is already in [Sto70a, Proposition 2]) Taking this together with the Conner–Floyd exact sequence of the pair $(\mathcal{A}, \mathcal{P})$ (see Remark 1.7.13) completes the proof. □

Together with Proposition 3.4.1 we immediately get the following.

Corollary 4.1.2. *The homomorphism*

$$\phi_{\mathfrak{N}} : \mathfrak{N}_*^G \to MO_*[e_V^{-1}, Y_{d,V}]$$

is a monomorphism if and only if $G = (\mathbb{Z}/2)^k$ for some k.

Next we want to illustrate the above proposition by citing a counterexample by Stong [Sto70b, p. 78]. He gives an example of a non-zero element in $\mathfrak{N}_3^{\mathbb{Z}/4}$ that is mapped to zero by $j_{\mathfrak{N}}$. See [Sto70b, p. 78]. Stong's element is represented by the $\mathbb{Z}/4$-manifold M defined as follows:

$$M := (D^3 \times \{0,1\})/\sim \quad \cong \mathbb{R}P^3 \amalg \mathbb{R}P^3.$$

Here D^3 denotes the disk in $\mathbb{R}^3$ with boundary S^2 and the equivalence relation $\sim$ on the product is given by $(x,k) \sim (-x,k)$ for $x \in S^2, k \in \{0,1\}$. The group $\mathbb{Z}/4 = \{0,1,2,3\}$ acts on M via

$$\begin{aligned} &\mathbb{Z}/4 \times M \to M \\ &(z,(x,k)) \mapsto ((-1)^{\lfloor \frac{z+k}{2} \rfloor} x, (-1)^z k + \frac{(-1)^{z+1}+1}{2}). \end{aligned}$$

(Note that $(-1)^z k + \frac{(-1)^{z+1}+1}{2}$ is nothing but the $\mod 2$ value of $z+k$.) The action is fixed point free, since the generator $1 \in \mathbb{Z}/4$ sends all points in one connected component $(D^3 \times \{0\})/\sim$ to points in the other connected component $(D^3 \times \{1\})/\sim$ and vice versa. So M is closed and without fixed points, hence

$$j_{\mathfrak{N}}([M]) = 0 \in \mathfrak{N}_3[\mathcal{A}, \mathcal{P}].$$

On the other hand, Stong proceeds to show that M is not null-bordant; see [Sto70b, p. 79].

4.2 Homotopical failure

The map $j_{MO} : MO_n^G[\mathcal{A}] \to MO_n^G[\mathcal{A}, \mathcal{P}]$ fails to be injective in general. We present an example of this failure, (compare [Han05, Section 4]).

Let W denote the 2-dimensional $\mathbb{Z}/4$-representation where we identify W with $\mathbb{C}$ and $\mathbb{Z}/4$ with the fourth roots of unity and let them act by multiplication. When we view W as a $\mathbb{Z}/2$ representation via the injective map $\mathbb{Z}/2 \to \mathbb{Z}/4$. It is isomorphic to $V \oplus V$, where V denotes the non-trivial one-dimensional representation of $\mathbb{Z}/2$. We form the $\mathbb{Z}/4$ bundle

$$\pi_2 : \; W \times \{-1,1\} \longrightarrow \{-1,1\},$$

where $\mathbb{Z}/4$ acts via

$$\begin{aligned} \mathbb{Z}/4 \times (W \times \{-1,1\}) &\to W \times \{-1,1\} \\ (z,(w,k)) &\mapsto (zw, z^2 k). \end{aligned}$$

This vector bundle is classified by a map

$$\{-1,1\} \to BO^{\mathbb{Z}/4}(2)$$

and this induces, together with the inclusion of 0 in W, a map

$$S^0 \times \{-1,1\} \to S^W \times \{-1,1\} \to TO^{\mathbb{Z}/4}(2),$$

which represents an element $x \in MO_{-2}^{\mathbb{Z}/4}$. Because the bundle $W \times \{-1,1\}$ is without fixed points, this element is mapped to zero by the geometric fixed point map: $\phi_{MO}(x) = 0$. But x is not the zero element. The injective map $\mathbb{Z}/2 \to \mathbb{Z}/4$ gives the restriction map:

$$\mathrm{res} : MO_{-2}^{\mathbb{Z}/4} \to MO_{-2}^{\mathbb{Z}/2}.$$

Since the $\mathbb{Z}/4$-bundle $W \times \{-1,1\}$ restricts to the $\mathbb{Z}/2$-bundle $(V \oplus V) \times \{-1,1\}$ over the trivial $\mathbb{Z}/2$-space $\{-1,1\}$, the map

$$S^0 \times \{-1,1\} \to S^{V \oplus V} \times \{-1,1\} \to TO^{\mathbb{Z}/2}(2)$$

represents the element $\mathrm{res}(x) = 2e_V^2 \in MO_{-2}^{\mathbb{Z}/2}$, which is non-zero. (Compare the description of $MO_*^{\mathbb{Z}/2}$ in [Sin02, Theorems 2.7 and 2.8]. In the notation used there, the element e_V^2 is $e_{2\sigma}$.) Together with Proposition 3.4.2 we get the following.

Corollary 4.2.1. *For* $G = \mathbb{Z}/4$

$$\iota \circ \phi_{MO} : MO_*^{\mathbb{Z}/4} \to MO_*[e_V, e_V^{-1}, Y_{d,V}]$$

is not a monomorphism.

5 Open Questions

In this chapter we collect a few questions, that one might ask in the context of the preceding chapters.

5.1 Complex equivariant cobordism

Looking at the results discussed in Section 2.3 the following questions come to mind.

Question 5.1.1 (compare Proposition 2.3.2). For which types of groups G is the complex equivariant Pontryagin–Thom map

$$\Omega_*^G \to MU_*^G$$

a monomorphism? And for what groups is it a *split* monomorphism?

Question 5.1.2 (compare Proposition 2.3.3). Is MU_*^G a free MU_*-module concentrated in even degrees for every compact Lie group G?

Comezaña conjectures a positive answer [Com96, p. 342] and claims to have verified the statement for the non-Abelian groups $O(2)$ and the dihedral groups. The author does not know of any counterexamples to the injectivity of the Pontryagin–Thom map, hence the following conjecture.

Conjecture 5.1.3. *The complex equivariant Pontryagin–Thom map*

$$PT : \Omega_*^G \to MU_*^G$$

is a monomorphism for all compact Lie groups G.

Comezaña's proof of the corresponding complex result for compact Abelian Lie group G relies on an induction on the cyclic factors of G. This cannot be done with non-Abelian G.

5.2 Real equivariant cobordism

In view of the explicit description of $\mathfrak{N}_*^{\mathbb{Z}/2}$, $MO_*^{\mathbb{Z}/2}$ and the quotient $MO_*^G/\mathfrak{N}_*^G$ in [Sin02, Theorems 2.4, 2.7 and 2.8] (also compare Section 2.2 and 3.6) we can ask the corresponding questions for $G = \mathbb{Z}/2 \times \cdots \times \mathbb{Z}/2$.

Question 5.2.1 (compare [Sin02, Theorem 2.4]). What are classes, such that MO_*^G is generated over $\mathfrak{N}_*^G$ (either additively or multiplicatively) by these classes and what is a complete set of relations?

Question 5.2.2 (compare [Sin02, Theorem 2.7]). By what classes is $\mathfrak{N}_*^G$ generated over $\mathfrak{N}_*^G$ (either additively or multiplicatively) as a sub-ring of MO_*^G?

Question 5.2.3 (compare [Sin02, Theorem 2.8]). How can the quotient $\mathfrak{N}_*^G$-module

$$MO_*^G/\mathfrak{N}_*^G$$

be described?

In view of our counterexamples we wonder in what respect Theorem 3.5.2 can fail.

Question 5.2.4. For what groups G is the diagram

$$\begin{tikzcd} \mathfrak{N}_*^G \arrow[r, "\phi_{\mathfrak{N}}"] \arrow[d, "PT"] & MO_*[e_V^{-1}, Y_{d,V}] \arrow[d, "i"] \\ MO_*^G \arrow[r, "\iota\circ\phi_{MO}"] & MO_*[e_V, e_V^{-1}, Y_{d,V}] \end{tikzcd}$$

commutative and a pull-back (even if the maps fail to be injective)?

Analogous to the complex case it is natural to ask the following question.

Question 5.2.5 (compare Question 5.1.2). For what groups G is MO_*^G a free MO_*-module?

In contrast to the corresponding complex case, already for the group $G = \mathbb{Z}/2$, a finite Abelian Lie group, MO_*^G is not a free MO_*-module. This can be seen from Sinha's description of $MO_*^{\mathbb{Z}/2}$ in [Sin02, Theorem 2.4]. Hence a real version of Proposition 2.3.3 cannot be expected and a proof like Comezaña's proof does not work in the real case.

The other question analogous to the complex case and another way in which Theorem 3.5.2 can fail is the following.

Question 5.2.6 (compare Proposition 2.3.2). For what groups G is the real equivariant Pontryagin–Thom map

$$PT : \mathfrak{N}_*^G \to MO_*^G$$

a monomorphism? And for what groups is it a *split* monomorphism?

Again Comezaña's proof of the corresponding complex result [Com96, p. 342] for compact Abelian Lie group G does not work here; it would also assume that MO_*^G is a free MO_*-module, which fails to be true in general in the real case (compare Question 5.2.5).

Not being concerned about the splitting, the question of injectivity remains. The result by tom Dieck (see Theorem 2.2.3 and [tD71, Theorem 2]) for $G = \mathbb{Z}/2 \times \cdots \times \mathbb{Z}/2$ relies on localization techniques, that only work for these groups; here Stong's result [Sto70b, Propositions 14.2 and 14.3] gives a definitive answer, (compare Proposition 4.1.1). So tom Dieck's proof cannot be easily adapted to more general G. Nonetheless one might conjecture (perhaps over-optimistically) a positive answer to Question 5.2.6 for all compact Lie groups G.

Appendix

List of categories

Whenever the homotopy categories of a category $\mathscr{C}$ is defined, it is denoted by $h\mathscr{C}$. Similarly whenever weak equivalences are defined, the resulting localized category $\mathscr{C}$ will be denoted by $\bar{h}\mathscr{C}$. For two objects X and Y in a category $\mathscr{C}$ we denote the set of morphisms between X and Y by $\mathscr{C}(X, Y)$.

$\mathcal{A}b$ the category of Abelian groups

$G\mathscr{P}\mathcal{U}$ the category of G-prespectra indexed on a G-universe $\mathcal{U}$

$G\mathscr{P}$ the category of G-prespectra (universe implicit)

$G\mathscr{T}$ the category of based G-spaces and G-maps

$G\mathscr{S}\mathcal{U}$ the category of G-spectra indexed on a G-universe $\mathcal{U}$

$G\mathscr{S}$ the category of G-spectra (universe implicit)

$G\mathscr{V}$ the category of G-vector bundles

$\mathscr{I}O(G,\mathcal{U})$ the category of indexing spaces in a G-universe $\mathcal{U}$

$\mathscr{I}O(G)$ the category of indexing spaces (universe implicit)

$\mathscr{S}$ the category of spectra

$\mathscr{T}$ the category of based compactly generated spaces and continuous maps

$\mathscr{U}$ the category of (unbased) compactly generated spaces and continuous maps

Ω-$G\mathscr{S}\mathcal{U}$ the category of Ω-G-spectra indexed on a G-universe $\mathcal{U}$

Ω-$G\mathscr{S}$ the category of Ω-G-spectra (universe implicit)

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