

THE COMPLETE ENUMERATION OF 4-POLYTOPES AND 3-SPHERES WITH NINE VERTICES

BY

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ABSTRACT

We describe an algorithm to enumerate polytopes. This algorithm is then implemented to give a complete classification of combinatorial spheres of dimension 3 with 9 vertices and decide polytopality of those spheres. In order to decide polytopality, we generate polytopes by adding suitable points to polytopes with less than 9 vertices and therefore realize as many as possible of the combinatorial spheres as polytopes. For the rest, we prove non-realizability with techniques from oriented matroid theory. This yields a complete enumeration of all combinatorial types of 4-dimensional polytopes with 9 vertices. It is shown that all of those combinatorial types are rational: They can be realized with rational coordinates. We find 316 014 combinatorial spheres on 9 vertices. Of those, 274 148 can be realized as the boundary complex of a four-dimensional polytope and the remaining 41 866 are non-polytopal.

1. Introduction

1.1. RESULTS. Having good examples (and counterexamples) is essential in discrete geometry. To this end, a substantial amount of work has been done on the classification of polytopes and combinatorial spheres; see Subsection 1.4. The

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classification of combinatorial 3-spheres and 4-polytopes for 7 vertices was done by Perles, [Grü67, Section 6.3] and for 8 vertices it was completed by Altshuler and Steinberg [AS85].

As a next step, we present new algorithmic techniques to obtain a complete classification of combinatorial 3-spheres with 9 vertices into polytopes and non-polytopes. We obtain the following results:

THEOREM 1: *There are precisely 316 014 combinatorial types of combinatorial 3-spheres with 9 vertices.*

THEOREM 2: *There are precisely 274 148 combinatorial types of 4-polytopes with 9 vertices.*

Therefore we have 41 866 non-polytopal combinatorial types of combinatorial 3-spheres with 9 vertices. By taking polar duals, we immediately also obtain a complete classification of 4-polytopes and 3-spheres with nine facets.

We provide rational coordinates for all of the combinatorial types of 4-polytopes with 9 vertices. We call a polytope **rational** if it is combinatorially equivalent to a polytope with rational coordinates.

COROLLARY 3: *Every 4-polytope with up to 9 vertices is rational. Every 4-polytope with up to 9 facets is rational.*

Perles showed, using Gale diagrams, that all d -polytopes with at most $d + 3$ vertices are rational; see [Grü67, Chapter 6]. There is an example of Perles of an 8-polytope with 12 vertices, which is not rational [Grü67, Thm. 5.5.4, p. 94]. For $d = 4$, there are examples of non-rational polytopes with 34 [RGZ95, Corollary of Main Theorem] and 33 vertices [RG06, Thm. 9.2.1].

Question 4: What is the smallest n , such that there is a non-rational 4-polytope with n vertices?

A list of all combinatorial 3-spheres with 9 vertices as well as rational polytopal realizations, if possible, or certificates for their non-polytopality is provided at <https://page.mi.fu-berlin.de/moritz/> and as ancillary files of the arxiv-version of this article [Fir18]. In Section 6, we summarize our results grouped by number of facets (Table 5) and by f -vector (Table 7).

1.2. METHODS. Enumerating all combinatorial 3-spheres and 4-polytopes is a challenging problem even for a relatively small number of vertices, not only because there is a huge number of them. In fact, already for 3-dimensional combinatorial spheres (see Definition 5), deciding whether a combinatorial sphere is polytopal is equivalent to the Existential Theory of the Reals [RGZ95, Corollary of Main Theorem]. However, for few vertices in small dimension a considerable amount of work has been done; for a summary see Subsection 1.4.

In order to enumerate all convex 4-polytopes with 9 vertices we proceed in three steps:

- (1) Completely enumerate combinatorial 3-spheres (Section 2).
- (2) Prove non-polytopality for some of them (Section 3).
- (3) Provide rational realizations for the rest of them (Section 4).

For the first step, we start with a set of simplicial spheres and repeatedly untriangulate them. This can be done by joining two facets in the face lattice. It then needs to be checked if the resulting face poset corresponds to a combinatorial sphere. For the second step, we resort to the theory of oriented matroids and use Graßmann–Plücker relations to obtain non-realizability certificates.

For the last step, instead of starting with a combinatorial sphere and deciding its realizability as a polytope, we present an algorithm to generate as many different combinatorial types as possible. We start with a complete set of realizations of combinatorial types of 4-polytopes with less than 9 vertices and inductively add the 9th point at strategic locations. These locations are carefully chosen by considering the hyperplane arrangement generated by the bounding hyperplanes of the polytopes with less than 9 vertices; compare [Grü67, Thm. 5.2.1.].

There is no reason why this method should generate all combinatorial types of 4-polytopes with 9 vertices, since it depends on the specific realizations of the polytopes with fewer vertices. This is treated in an exercises by Grünbaum [Grü67, Ex. 5.2.1]. In [Grü67, Section 5.5] he explains:

[...] if we are presented with a finite set of polytopes it is possible to find those among them which are of the same combinatorial type. It may seem that this fact, together with theorem 5.2.1 which determines all the polytopes obtainable as convex hulls of a given polytope and one additional point, are sufficient to furnish an *enumeration* of combinatorial types

of d -polytopes. [...] However, from the result of exercise 5.2.1 it follows that it may be necessary to use different representatives of a given combinatorial type in order to obtain all the polytopes having one vertex more which are obtainable from polytopes of the given combinatorial type. Therefore it is not possible to carry out the inductive determination of all the combinatorial types in the fashion suggested above.

We agree that in general “not possible” to determine all combinatorial types inductively. This is obvious in the presence of non-rational combinatorial types, which can never be generated inductively in this fashion. However, for a small number (we show less than 10) of vertices and $d = 4$ it turns out that this approach is sufficient to determine all combinatorial types.

For each of the steps above we need to be able to quickly decide if a given combinatorial sphere of polytopes has been generated before. For this it is enough to look at the vertex-facet incidences of the corresponding face lattice. Therefore, given two of those face lattices it is sufficient to check if the two directed vertex-facet graphs are isomorphic. For this we employ tools from Graph Canonization: In order to check if a combinatorial type is already contained in a set \mathcal{S} of N combinatorial types, we do not run graph isomorphism N times. Instead, we precompute a (hashable) canonical form for all of the graphs in the set. Then we can simply check if the canonical form of the vertex-facet graph of the given combinatorial type is in the set of normal forms of graphs in \mathcal{S} . After computing the canonical form, the average case to check if a graph is in \mathcal{S} will take constant time.

For previous work on using graph canonization to enumerate polytopes, see [KS03, p. 216], [BSP⁺14] and [JPS19, Sec. 3].

Hardware and computing time. The computations in Section 2 were performed in about 10 hours on a single desktop computer with 8 cores (Intel i7-7700) running at 3.6GHz with 32GB RAM. The computations for Sections 3 and 4 were performed on the *allegro* cluster at FU-Berlin, which has about 1000 cores (Intel Xeon, mainly X5650) running at 2.6GHz and having about 3.5TB combined RAM and runs on linux. The results from Section 3 were obtained in about 800 CPU-hours and the results from Section 4 in about 2000 CPU-hours. We used sagemath [SD18] to implement the algorithms described below.

1.3. DEFINITIONS. We assume basic familiarity with convex polytopes; see [Grü67] and [Zie95] for comprehensive introductions. For Section 3, we assume familiarity with the basic notions of the theory of oriented matroids, especially in the guise of chirotopes; here the standard references include [BLVS⁺99], [RGZ04, Sect. 6] and [BS89].

Simplicial spheres, that is, simplicial complexes homeomorphic to a sphere, arise as boundaries of simplicial polytopes. The notion combinatorial spheres is sometimes interchangeable with such “simplicial spheres”, but we use it in a more general way and give a concise definition below. The intention of the definition is to get a set of combinatorial spheres that fits into the following diagram:

$$\begin{array}{ccc} \text{simplicial polytopes} & \subset & \text{polytopes} \\ & \cap & \cap \\ \text{simplicial spheres} & \subset & \text{combinatorial spheres} \end{array}$$

Here vertical inclusions indicate “taking boundary” of the polytopes.

Definition 5 (combinatorial sphere, equivalent to [BZ18, Def. 2.1]): For $d \in \mathbb{N}$, a **strongly regular** d -cell complex \mathcal{C} is a finite d -dimensional CW -complex, that is, a collection of k -cells for $k \leq d$, such that the following two properties hold:

- (1) **regularity**: the attaching maps of all cells are homeomorphisms also on the boundary.
- (2) **intersection**: the intersection of two cells in \mathcal{C} is again a cell in \mathcal{C} (possibly empty).

A strongly regular d -cell complex is called a **combinatorial sphere** if it is homeomorphic to S^d .

It follows that for each k -cell F , there is a k -polytope $P(F)$ and a homeomorphism h_F from F to H , such that the preimages of the faces of $P(F)$ under h_F are again cells of \mathcal{C} ; compare [AS84, Section 2] and [Bar73, Section 2].

Definition 6 (Eulerian and interval connected; compare [BZ18, Def. 2.1]): A finite graded lattice of rank d is called

- **Eulerian** if all non-trivial closed intervals have the same number of odd and even elements and it is
- **interval connected** if all open intervals of length at least 3 are connected.

The boundary of a d -polytope gives rise to a combinatorial $(d - 1)$ -sphere. The intersection poset of the set of cells of a combinatorial sphere is an Eulerian lattice, which is interval connected. Two polytopes (or two combinatorial spheres) are called **combinatorially equivalent** if they give rise to isomorphic face lattices. All properties, that only depend on the isomorphism type of face lattice, such as (flag) f -vector, are well defined for combinatorial types of polytopes (or combinatorial spheres).

PROPOSITION 7: *For $d = 4$, every interval connected Eulerian lattice of rank $d + 1$ is the face lattice of a strongly regular $(d - 1)$ -cell complex.*

For a proof we refer the reader to [BZ18, Prop. 2.2].

For $d = 4$ it is therefore possible to describe a combinatorial sphere purely in combinatorial terms and it is sufficient to characterize the sphere completely by a set of facets, each containing a set of vertices. This is how we will describe combinatorial spheres below.

1.4. PREVIOUS RESULTS. Classifications of $(d - 1)$ -spheres and d -polytopes with n vertices have been obtained for various dimensions d and number of vertices n . Also certain subfamilies of all such spheres and polytopes, namely simplicial and neighborly ones have been classified.

In dimension 3, Steinitz' theorem, [Ste22, Satz 43, p. 77] reduces the classification 3-polytopes to the classification of planar 3-connected graphs. There are results on the asymptotic behavior; see [BW88], [Tut80], [RW82] and [Slo, A944].

If the number of vertices n is less than or equal to $d + 3$, then every combinatorial $(d - 1)$ -sphere coincides with the number of d -dimensional polytopes and explicit formulas are known; see [Fus06, Thm. 1] and [Grü67, Sect. 6.1]. The techniques used for these results are Gale-diagrams. In dimension 4, we summarize known results in Table 1. In each case, the paper we cite is the one that completes the classification; in some cases the complete classification was done in a series of papers.

While various classifications of simplicial and neighborly spheres and polytopes have been obtained in the meantime, the latest classification of all 3-spheres and 4-polytopes was completed by Altshuler and Steinberg for 8 vertices in [AS85]. Among other methods, they consider which 3-polytopes can

Table 1. Classification results for 4-polytopes with ≤ 11 vertices. Boldface results are new.

# of vertices	5	6	7	8	9	10	11
<i>all</i>							
# of f -vectors	1	4	15	40	88	?	?
3-spheres	1	4	31	[AS85] 1 336	316 014	?	?
4-polytopes ¹	1	4	31	[AS85] 1 294	274 148	?	?
non-polytopal	0	0	0	42	41 866	?	?
<i>simplicial</i>							
# of f -vectors	1	2	4	7	11	16	22
3-spheres	1	2	5	[Bar73] 39	[AS76] 1 296	[Lut08] 247 882	[SL09] 166 564 303
4-polytopes ²	1	2	5	[GS67] 37	[ABS80] 1 142	[Fir17] 162 004	?
non-polytopal	0	0	0	2	154	85 878	?
<i>neighborly</i>							
3-spheres	1	1	1	[GS67] 4	[AS74] 50	[Alt77] 3 540	[SL09] 897 819
4-polytopes ³	1	1	1	[GS67] 3	[AS73] 23	[BS87] 431	[Fir17] 13 935
non-polytopal	0	0	0	1	27	3 109	883 884

appear as facets of a 4-polytope. In our present classification for 9 vertices we use completely different algorithmic methods.

Recently, Brinkmann and Ziegler enumerated all combinatorial 3-spheres with f -vector (f_0, f_1, f_2, f_3) , such that $f_0 + f_3 \leq 22$; see [BZ18, Table 1]. This includes all combinatorial 3-spheres on 9 vertices up to 13 facets. On the other hand, there has been an earlier attempt by Engel to classify all combinatorial 3 spheres with 9 vertices; see [Eng91, Table 6]. The results of Brinkmann and Ziegler contradict the results of Engel for the number of combinatorial 3-spheres with f -vector $(9, f_1, f_2, k)$ for $k \in \{10, 11, 12, 13\}$. Because of this disagreement, it is desirable to have an independent check of the result. We provide this with our results in Section 2. Our classification below partially agrees with the results of Brinkmann and Ziegler (for $k < 10$ and $k \leq 12$) and partially with those of Engel (for $k < 10$ and $k > 13$). We explain this in detail at the end of Section 2.

¹ [Slo, A5841].

² [Slo, A222318].

³ [Slo, A133338].

2. Generating combinatorial spheres

We generate a complete set of combinatorial 3-spheres with n vertices from a complete set of simplicial 3-spheres with n vertices; for each sphere in this set, we generate all spheres obtained by untriangulating. By this we mean constructing a new combinatorial sphere from an old one by removing a ridge, that is, a 1-dimensional face. A combinatorial sphere is determined by its face lattice and the face lattice can be completely recovered from the incidence of the atoms and coatoms, that is, from the vertex-facet graph. We consider a combinatorial sphere M as the set of its facets, each facet being a set of vertices.

Definition 8: Let M be a combinatorial 3-sphere given as a set of facets, where each facet is a given set of vertices. Let $f_1, f_2 \in M$ be two of its facets that intersect in a ridge

$$r = f_1 \cap f_2.$$

Then the **untriangulation of M with f_1 and f_2** is the set $U(M)$, obtained from M by replacing f_1 and f_2 by their union:

$$U(M) := \{f_1 \cup f_2\} \cup M \setminus \{f_1, f_2\}.$$

The untriangulation $U(M)$ might not correspond to a combinatorial sphere. For example, if the new face $f_1 \cup f_2$ completely contains another face of $U(M)$ or if there is a face in M that intersects both f_1 and f_2 but does not intersect r , then $U(M)$ will not be a combinatorial sphere. In our procedure after generating $U(M)$ it remains to be checked if the corresponding face poset $P(M)$ is graded of rank $3 + 1$, Eulerian and strongly connected.

Since every combinatorial sphere can be triangulated (not necessarily in a unique way) until it is a simplicial sphere, all combinatorial spheres can be obtained by repeatedly untriangulating simplicial spheres. In the process of iteratively untriangulating, we might encounter a combinatorial type of combinatorial spheres multiple times. To detect this, we store a canonical form of the (directed) vertex-facet graph, just as we do in Section 4. For the same reason we keep a set of such graphs of combinatorial types of posets that do not correspond to combinatorial spheres. If we get such a type when untriangulating, we don't need to untriangulate any further.

This iteration process can be done in multiple ways. We present Algorithm 1, a recursive formulation, and remark:

Algorithm 1 Enumerating combinatorial spheres

Input: A dictionary *SimpSpheres* of all *simplicial* $(d - 1)$ -spheres with n vertices with key-values pairs (G, S) , where S is a set of facets, each facet containing a subset of the vertices $\{1, \dots, n\}$ and G is a canonical form of the vertex-facet graph of S .

Output: A dictionary *CombTypes* of all *combinatorial* spheres with n vertices. The dictionary *CombTypes* is of the same form as the dictionary *SimpSpheres* given as input.

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1: procedure UNTRIANGULATE( $M, \text{CombTypes}, \mathcal{N}\text{onTypes}$ ) ▷ ... recursively
2:    $G \leftarrow$  canonical form of vertex-facet graph of  $M$ 
3:   if ( $G$  is key of CombTypes) or ( $G$  in  $\mathcal{N}\text{onTypes}$ ) then
4:     return CombTypes,  $\mathcal{N}\text{onTypes}$ 
5:   else
6:      $P \leftarrow$  poset of  $M$ 
7:     if  $P$  is graded of rank  $d + 1$ , Eulerian, strongly connected then
8:       CombTypes[ $G$ ]  $\leftarrow M$ 
9:       for  $f_1, f_2 \in \binom{M}{2}$  do
10:        if  $f_1 \cap f_2$  is a ridge in  $P$  then
11:           $U(M) \leftarrow \{f_1 \cup f_2\} \cup M \setminus \{f_1, f_2\}$ 
12:          CombTypes,  $\mathcal{N}\text{onTypes} \leftarrow$  UNTRIANGULATE( $U(M), \text{CombTypes}, \mathcal{N}\text{onTypes}$ )
13:        end if
14:      end for
15:    else
16:       $\mathcal{N}\text{onTypes} \leftarrow \mathcal{N}\text{onTypes} \cup \{G\}$ 
17:    end if
18:  end if
19:  return CombTypes,  $\mathcal{N}\text{onTypes}$ 
20: end procedure

21: CombTypes  $\leftarrow$  empty dictionary
22:  $\mathcal{N}\text{onTypes} \leftarrow$  empty set
23: for  $S \in \text{SimpSpheres}$  do
24:   CombTypes,  $\mathcal{N}\text{onTypes} \leftarrow$  UNTRIANGULATE( $S, \text{CombTypes}, \mathcal{N}\text{onTypes}$ )
25: end for
26: return CombTypes

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LINE 2: Canonical form of vertex-facet graph of M : This can be computed by using *bliss*, [JK15]; compare Algorithm 2 in Section 4.

LINE 6: $P \leftarrow$ poset of M : We compute the poset by iteratively calculating the intersection of the facets.

LINE 10/LINE 11: additional checks if $U(M)$ can possibly be the set of facets of a combinatorial sphere that can be added here.

LINE 12: the recursion: We know that the recursion terminates, since $U(M)$ has always strictly less elements (facets) than M .

LINE 9/LINE 23: These loops can be parallelized, when keeping multiple copies of *CombTypes* and $\mathcal{N}\text{onTypes}$ and merging them afterwards.

We use an implementation of Algorithm 1 to generate all combinatorial 3-spheres with up to 9 vertices. We start from simplicial 3-spheres: those have been enumerated up to 10 vertices; see [Lut08] and for $n \leq 9$, we use the tables provided by Lutz, [Lut]. The enumeration of the relevant 1296 simplicial spheres with 9 has been completed by Altshuler and Steinberg, [AS76]. Since all combinatorial spheres can be obtained by recursively untriangulating, we obtain

THEOREM 9: *There are precisely 316 014 combinatorial types of combinatorial 3-spheres with 9 vertices.*

The method explained above could be summarized as “flattening a ridge”. The dual operation would be “edge-reduction”, that is, shrinking an edge until two vertices coincide. This is described by Engel (also in the case of 3-dimensional polytopes); see [Eng91, Section 2] and [Eng82]. In fact, in [Eng91] an enumeration of all combinatorial 3-spheres with 9 facets is attempted. However, our results partially disagree with those of Engel. We translate the results of [Eng91, Table 6] in the dual setting; then for each number of facets $5 \leq k \leq 27$ we find a number of combinatorial 3-spheres with 9 vertices. We compare these numbers to our Table 5 and find that the numbers agree for all $k \in \{6, 5, 8, 9\}$ and $14 \leq k \leq 27$, but not for $k \in \{10, 11, 12, 13\}$. In all of those cases, Engel claims to have found more combinatorial types of spheres. He does not provide a list of spheres but only their count; therefore we cannot show if he might have counted some of the spheres twice. This might have been the case, because he uses an ad hoc method to determine whether two combinatorial spheres are isomorphic [Eng91, Section 3], while we reduce the problem to checking if two (directed) graphs are isomorphic.

For up to 13 facets an enumeration of 3-spheres with 9 vertices is done by Brinkmann and Ziegler [BZ18]. Our results agree with theirs for all k except $k = 13$. In their paper yet a different method for generating combinatorial spheres is used. They start by generating all possible vertex-edge graphs of possible spheres and then sorting out those that are non-spheres; see [BZ18, Algorithm 3.1]. While this approach is valid, there seems to have been some problem with the implementation, leading to the inconsistency with our results. However there is an inconsistency only for f -vectors with $k = 13$ facets and only for 2 out of the 6 f -vectors with $k = f_3 = 13$. This is the largest number of facets Brinkmann and Ziegler consider. Brinkmann and Ziegler acknowledge

that their results for $k = 13$ are false and that the additional combinatorial spheres provided here are indeed not isomorphic to the one they have found. For all other f -vectors our results agree.

In Table 2, we give the f -vectors and counts of those cases where our results differ from [BZ18] or [Eng91]. Both papers do not attempt to completely decide which of the combinatorial spheres are in fact boundary of polytopes as we do in Section 4.

Table 2. Inconsistent results for the number of combinatorial types of 3-spheres with 9 vertices for some f -vectors. For all other f -vectors the numbers agree.

f -vector	[BZ18, Table 1]	Table 5	[Eng91, Table 5]
$(9, *, *, 9)$	1905	1905	1908
$(9, *, *, 10)$	5376	5376	5411
$(9, *, *, 11)$	11825	11825	11974
$(9, *, *, 12)$	20975	20975	21129
$(9, 28, 32, 13)$	2136	2224	not listed
$(9, 29, 33, 13)$	27	45	not listed
$(9, *, *, 13)$	20871 ¹	20975	21129

3. Proving non-polytopality

Proving non-polytopality of combinatorial spheres, and more generally the non-realizability of oriented matroids, has been done in an ad-hoc fashion (see the earlier classification results in Subsection 1.4 and Table 1) until the introduction of biquadratic final polynomials in [BR90]. Since then the language of oriented matroids has been sufficiently developed to phrase non-realizability results in that language. Oriented matroids can be effectively enumerated using SAT-solvers, as pioneered by Schewe, [Sch10]. An overview of results using oriented matroids to prove non-realizability as well the technique of partial oriented matroids used below can be found in [Fir17].

¹ This number is obtained by summing all f -vector of the form $(9, *, *, 13)$:
 $33 + 1223 + 7677 + 9773 + 2136 + 27$

In order to prove that some of the combinatorial 3-spheres obtained in Section 2 are not polytopal, we analyze the orientation information which can be deduced from the combinatorial spheres. Let us consider a combinatorial 3-sphere P , which is realized by the boundary of a 4-polytope. Then every ordered set of 5 vertices of P can be assigned a sign $\{-1, 0, 1\}$ depending on whether it spans a simplex of negative, zero or positive (signed) volume. The theory of oriented matroids abstracts these concepts and collects the orientation information in the chirotope map χ ; see [BLVS⁺99, Chapter 3.6] for a detailed introduction. The following three rules are satisfied by the boundary of a 4-polytope. To state them, we only need the incidence data, therefore they can be used on combinatorial spheres.

- (1) If five vertices a, b, c, d, e lie in a common facet, then $\chi(a, b, c, d, e) = 0$.
- (2) If four vertices a, b, c, d lie in a common facet, then for every pair e, e' outside of that facet we have

$$\chi(a, b, c, d, e) = \chi(a, b, c, d, e').$$

- (3) If the three vertices a, b, c lie in a common ridge R , which has the two adjacent facets F and F' , such that $R = F \cap F'$, then for every pair $d \in F, d' \in F'$ and all e not in F and not in F' we have

$$\chi(a, b, c, d, e) = -\chi(a, b, c, d', e).$$

Given a combinatorial 3-sphere S (as a set of facets), we first find a **partial chirotope**, which can be constructed using the rules above. That is, we find subset of $s_0 \subset \binom{\text{Vertices of } S}{5}$ and a map $\chi: s_0 \rightarrow \{-1, 0, 1\}$. This can be done first adding all the signs from rule (1), fixing a non-zero sign for an instance of rule (3) and then greedily applying rules (2) and (3) repeatedly. Apart from the choice of the first sign, there is no other choice for the signs defined by χ . Then we can check the **Graßmann–Plücker relations**, which need to be satisfied. The Graßmann–Plücker relations involve 6 values of the map χ . Of course, with the partial chirotope on s_0 we can only check those Graßmann–Plücker relations where all of those values are defined. If a Graßmann–Plücker relation is violated we can conclude that the sphere in question is not polytopal.

LEMMA 9: *Out of the 316 014 combinatorial 3-spheres with 9 vertices, there are 24 028 spheres, which give rise to a partial chirotope on s_0 , which contradicts a Graßmann–Plücker relation and are therefore not polytopal.*

Proof. For each of the 3-spheres we provide the corresponding s_0 , the partial chirotope and a violating Graßmann–Plücker relation. ■

In a next step we seek to enlarge the set s_0 , where a partial chirotope can be defined. To this end, we consider Graßmann–Plücker relations with 5 elements from s_0 , where the partial chirotope is already defined, and one element from $\binom{\text{Vertices of } S}{5}$, where the partial chirotope is not yet known. In some cases we can determine the sign of the new element, add it to s_0 and repeat. Iterating this can lead to a contradiction if the combinatorial sphere is not polytopal.

LEMMA 10: *Out of the 316 014 combinatorial 3-spheres with 9 vertices, there are 17 755 spheres, for which a contradiction arises when completing the partial chirotope on s_0 . Therefore those 17 755 spheres are not polytopal.*

Proof. For each of the 3-spheres we provide the corresponding s_0 , the chirotope together with a finite list of deductions, each expanding the definition of the partial chirotope to a new element using the Graßmann–Plücker relations until a contradiction is reached. ■

In some cases using the method of Lemma 10 does not lead to a contradiction after a finite number of steps. We then have a partial chirotope χ on a set s_1 , which contains s_0 and which cannot be enlarged by the steps described above. In some cases s_1 might be a complete chirotope.

LEMMA 11: *Out of the 316 014 combinatorial spheres of 3-spheres with 9 vertices, there are 83 spheres, for which the partial chirotope s_0 is completed to a chirotope on s_1 , which admits a biquadratic final polynomial. Therefore those 83 spheres are not polytopal.*

Proof. For all the relevant 83 cases, it turns out that s_1 is actually a complete chirotope. In each case, we provide the completed chirotope together with the unfeasible linear program associated to the biquadratic final polynomial. ■

Only 11 of the 83 cases already admit a biquadratic final polynomial for the partial chirotope on the set s_0 . For the other cases there is no biquadratic final polynomial on the the partial chirotope and we need to complete the chirotope in order to prove non-polytopality.

Since the sets of non-polytopal spheres in Lemma 9, Lemma 10 and Lemma 11 are disjoint, we obtain

THEOREM 12: *There are at most*

$$316\,014 - 24\,028 - 17\,755 - 83 = 274\,148$$

4-polytopes with 9 vertices.

4. Generating combinatorial types of polytopes

We describe an algorithm to generate combinatorial types of polytopes. Let Q be a d -polytope with n vertices and k facets, let $\mathcal{H}(Q)$ denote the affine hyperplane arrangement consisting of the k hyperplanes supporting the facets of Q . We view a supporting hyperplane h as an element in $\mathbb{R} \times \mathbb{R}^d$, associated to the hyperplane containing $x \in \mathbb{R}^d$ if and only if their dot product is zero: $(1, x) \cdot h = 0$. The faces of the hyperplane arrangement are given by

$$\text{faces}(\mathcal{H}(Q)) := \{F_\alpha \subset \mathbb{R}^d \mid \alpha = (\alpha_1, \alpha_2, \dots, \alpha_k) \in \{-1, 0, 1\}^k \text{ if } F_\alpha \neq \emptyset\},$$

where

$$F_\alpha = \left\{ x \in \mathbb{R}^d \mid \begin{array}{ll} (1, x) \cdot h_i \leq 0 & \text{if } \alpha_i \in \{-1, 0\} \\ (1, x) \cdot h_i \geq 0 & \text{if } \alpha_i \in \{0, 1\} \end{array} \text{ for } 1 \leq i \leq k \right\}$$

By definition, the relative interiors of the faces partition \mathbb{R}^d :

$$\mathbb{R}^d = \bigcup_{F \in \text{faces}(\mathcal{H}(Q))} \overset{\circ}{\text{relint}}(F).$$

PROPOSITION 13: *Let Q be a d -polytope and $F \in \text{faces}(\mathcal{H}_j(Q))$ a face in the hyperplane arrangement of its supporting hyperplanes. Then for any two points $q_1, q_2 \in \text{relint}(F)$ in the relative interior of F , the polytopes*

$$Q_i := \text{conv}(Q \cup \{q_i\})$$

for $i = 1, 2$ are combinatorially equivalent.

The proposition is a reformulation of [Grü67, Thm. 5.2.1] and a proof can be found there.

Motivated by Proposition 13, we proceed inductively to generate combinatorial types of d -polytopes with k vertices, starting from a set of polytopes with $k - 1$ vertices. Given a polytope Q with $k - 1$ vertices, we choose an interior point p from each face of $\mathcal{H}(Q)$. Then we form the convex hull of $Q \cup \{p\}$ and check if this yields a polytope with k vertices. If this is the case, we check if

we have seen the combinatorial type of this polytope before. If not, we add it to our output. In order to check quickly if we have already found a combinatorial type, we calculate a canonical form of the (directed) vertex-facet graph, which depends only on the isomorphism class of the graph and therefore only on the combinatorial type of the polytope: it is possible to recover the entire face lattice from the vertex-facet graph. The canonical form can then be used in a hash table or a dictionary.

Algorithm 2 Generating polytopes

Input: An integer k and a set of polytopes with $k - 1$ vertices \mathcal{Q}

Output: A dictionary \mathcal{P} with key-value pairs (G, P) , where P is a polytope with k vertices and G is a canonical form of the vertex-facet graph of P .

```

1: procedure UPDATE( $\mathcal{P}, P$ )
2:    $G \leftarrow$  canonical form of vertex-facet graph of  $P$ 
3:   if  $G$  is not key of  $\mathcal{P}$  then
4:      $\mathcal{P}[G] \leftarrow P$ 
5:   end if
6: end procedure

7:  $\mathcal{P} \leftarrow$  empty dictionary
8: for  $Q \in \mathcal{Q}$  do
9:   for  $F \in$  faces( $\mathcal{H}(Q)$ ) do
10:     $p \leftarrow$  interior point of  $F$        $\triangleright$  different choices are possible, e.g. center of (bounded)  $F$ 
11:     $P \leftarrow$  conv( $Q \cup \{p\}$ )
12:    if number of vertices of  $P = k$  then
13:      UPDATE( $\mathcal{P}, P$ )
14:    end if
15:   end for
16: end for
17: return  $\mathcal{P}$ 

```

This is all summarized in Algorithm 2. Let us explain some details of this algorithm:

LINE 2: canonical form of vertex-facet graph of P : This can be computed by using *bliss* [JK15].

LINE 3: if G is already a key in \mathcal{P} , that is, there is already a polytope P' , which is combinatorially isomorphic to P in the dictionary \mathcal{P} , then we could still decide to update the dictionary; for example, if P has a shorter description than P' , i.e., simpler rational coordinates.

LINE 8: $Q \in \mathcal{Q}$: at this point the algorithm can be parallelized; each case Q can be run separately yielding a dictionary \mathcal{P}_Q . Those must then be collected to give the desired dictionary \mathcal{P} .

LINE 9: $F \in \text{faces}(\mathcal{H}(Q))$: the hyperplane arrangement can be computed using *sagemath* methods [SD18].

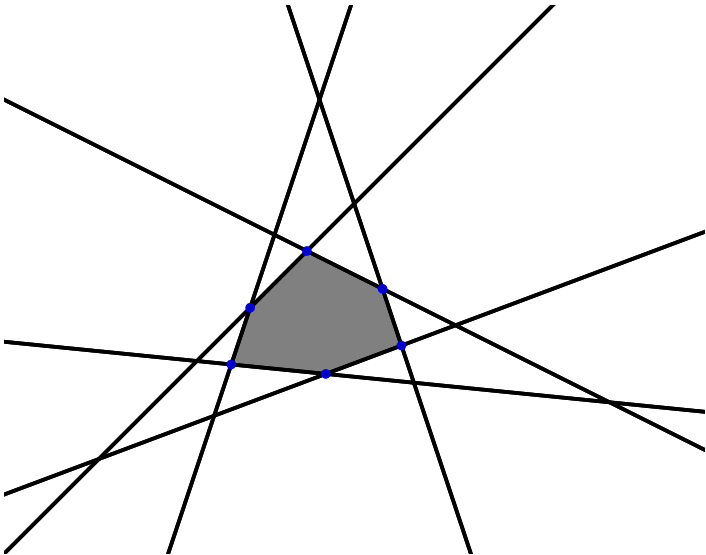
LINE 10: $p \leftarrow$ interior point of F : Here we have some choice for interior point of the rational, not necessarily a bounded polyhedron, which is a face of the hyperplane arrangement F . To avoid dealing with unbounded polyhedra, we intersect the entire hyperplane arrangement $\mathcal{H}(Q)$ with a rational cuboid (for example, axes aligned) that contains all of the vertices of $\mathcal{H}(Q)$ in its interior. The vertices of $\mathcal{H}(Q)$ are its zero-dimensional faces. In general, it has more vertices than Q . This reduces the problem to finding a rational point in the relative interior of a rational polytope. We can simply take the barycenter of the vertices of the polyhedron.

We illustrate the procedure in Figure 1 by looking at what happens to an irregular hexagon. Since the classification of polytopes in dimension 2 is so simple, it might seem like a wasteful way to generate a heptagon, but in higher dimensions the classifications get more interesting.

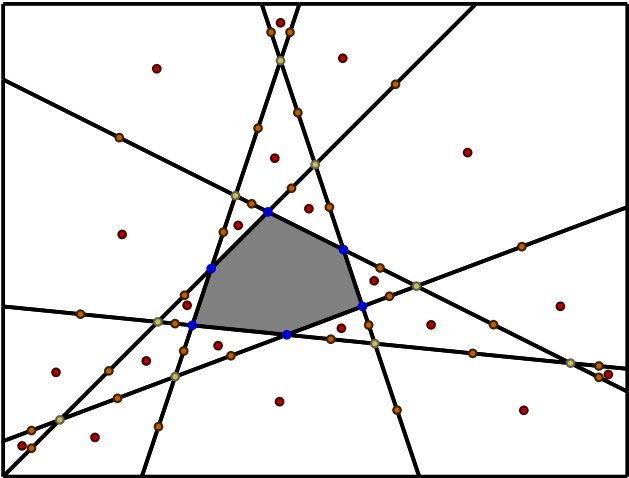
We use an implementation of Algorithm 2 for the generation of 4-polytopes. We start with a realization of the simplex comprised of the origin together with the standard basis vectors of \mathbb{R}^4 . In running Algorithm 2, there is some choice involved: in Line 10, we choose a point in the interior of a (potentially unbounded) polyhedron.

In a first run, we intersect the unbounded polyhedra with an axes aligned cuboid, which contains all the vertices of $\mathcal{H}(Q)$ with a padding of 1 unit. For example, when going from the simplex in the first step to polytopes with 6 vertices, we intersect the cells in the hyperplane arrangement with the axes aligned cuboid given by the two coordinates $(-1, -1, -1, -1)$ and $(2, 2, 2, 2)$. Then we choose as an interior point the barycenter.

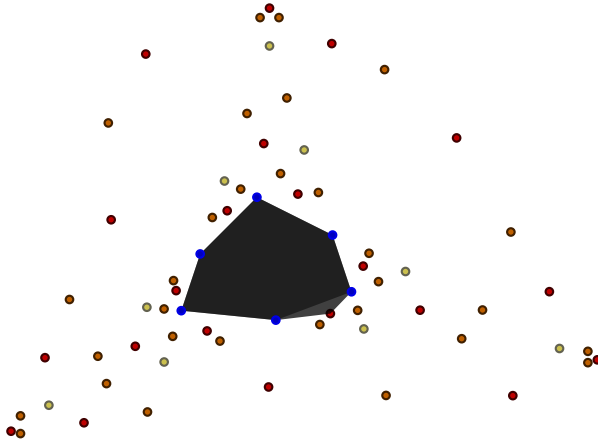
In a second run, we choose the same bounding cuboid, but choose the interior point of the bounded polyhedra differently: we strive for comparatively ‘simple’ rational coordinates. For example, we might look for rational numbers with small absolute values for numerator and denominator. (It is of course conceivable to take another definition of ‘simple’.) We pick a subset of vertices from the set of all vertices of the polyhedron that affinely span the affine hull of the polyhedron. Then we look at the barycenter of this subset of vertices and choose the subset with the ‘simplest’ rational coordinates.



(a) The hyperplane arrangement induced by the facets...



(b) ... in a bounding box with barycenters of all faces.



Adding a point gives a new combinatorial type.

Figure 1. Generating a heptagon from a hexagon.

Putting together the results from these two runs, we obtain

THEOREM 14: *There are at least 274 148 combinatorial types of 4-polytopes with 9 vertices.*

Proof. We provide rational coordinates for all combinatorial types in question. ■

This theorem together with Theorem 12 implies

THEOREM 15: *There are precisely 274 148 combinatorial types of 4-polytopes with 9 vertices.*

5. Applications

The complete classification of combinatorial 3-spheres and 4-polytopes with up to 9 vertices immediately has some applications. We only want to provide two such applications here.

5.1. NON-REALIZABLE FLAG f -VECTORS. Recently, Brinkmann and Ziegler provided the first example of a flag f -vector of a combinatorial sphere that does not appear as the flag f -vector of a polytope, [BZ17]. Such a flag f -vector is called **non-realizable**. The non-realizable flag f -vector they provide is $(f_0, f_1, f_2, f_3; f_{02}) = (12, 40, 40, 12; 120)$. There is precisely one sphere, but no polytope with this flag f -vector. Our complete classification gives three additional examples of non-realizable flag f -vectors. The non-realizable flag

f -vectors are $(9, 25, 26, 10; 50)$, $(9, 27, 29, 11; 53)$ and $(9, 27, 30, 12; 57)$. For the last two we have two types of combinatorial spheres and for the first one there is a unique type of combinatorial sphere. We give the sphere as a list of facets in Table 3. (Here vertices are the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and a facet 12345 is an abbreviation for the set $\{1, 2, 3, 4, 5\}$.)

Table 3. Non-realizable flag f -vectors and spheres with those flag f -vectors

flag f -vector	facets of non-realizable 3-sphere
$(9, 25, 26, 10; 50)$	[12345,12469,12578,12678,13468, 1358,23459,25679,346789,35789]
$(9, 27, 29, 11; 53)$	[12346,12357,12678,1345,14568, 15789,2349,23579,24679,34589,46789]
$(9, 27, 29, 11; 53)$	[12345,12469,12567,13468,13578, 1678,23489,2359,25679,35789,46789]
$(9, 27, 30, 12; 57)$	[12345,12468,12567,13458,15789, 16789,23479,2357,24679,34689,3579,3589]
$(9, 27, 30, 12; 57)$	[1234,12358,1246,12567,13468, 15789,16789,23457,24679,34579,34689,3589]

5.2. VERTEX-EDGES GRAPHS OF POLYTOPES. In his PhD thesis [Esp14], Espenschied examines under what circumstances the complete t -partite graph K_{n_1, n_2, \dots, n_t} can appear as the vertex-edge graph of a polytope.

CONJECTURE 15 (Espenschied’s conjecture [Esp14, p. 82]): *If K_{n_1, n_2, \dots, n_t} is the graph of a polytope, then $\{n_1, n_2, \dots, n_t\} \subset \{1, 2\}$ as sets.*

We disprove the above conjecture by looking at the graphs of all 4-polytopes with 9 vertices and find a number of counter-examples to this conjecture. In fact, there are 14 polytopes that contradict Espenschied’s conjecture. We list the complete multipartite graphs and the number of combinatorial types of polytopes with that graph in Table 4. See the survey paper by Bayer [Bay18] for more on graphs of polytopes.

Table 4. Number of counter-examples to Espenschied’s conjecture

Graph	$K_{3,2,2,2}$	$K_{3,2,2,1,1}$	$K_{3,2,1,1,1,1}$	$K_{3,1,1,1,1,1,1}$
number of combinatorial types	1	2	5	6

6. Tables of results

Table 5. Combinatorial 3-spheres and 4-polytopes with ≤ 9 vertices, grouped by number of facets

<i>f</i> -vector	3-spheres	4-polytopes	non-realizable	# of <i>f</i> -vectors	<i>f</i> -vector	3-spheres	4-polytopes	non-realizable	# of <i>f</i> -vectors
(5, *, *, 5)	1	1	0	1	(9, *, *, 6)	1	1	0	1
(6, *, *, 6)	1	1	0	1	(9, *, *, 7)	7	7	0	2
(6, *, *, 7)	1	1	0	1	(9, *, *, 8)	76	76	0	4
(6, *, *, 8)	1	1	0	1	(9, *, *, 9)	467	463	4	6
(6, *, *, 9)	1	1	0	1	(9, *, *, 10)	1905	1872	33	5
(6, *, *, *)	4	4	0	4	(9, *, *, 11)	5376	5218	158	6
(7, *, *, 6)	1	1	0	1	(9, *, *, 12)	11825	11277	548	6
(7, *, *, 7)	3	3	0	2	(9, *, *, 13)	20975	19666	1309	6
(7, *, *, 8)	5	5	0	2	(9, *, *, 14)	31234	28821	2413	5
(7, *, *, 9)	7	7	0	2	(9, *, *, 15)	39875	36105	3770	6
(7, *, *, 10)	6	6	0	2	(9, *, *, 16)	44461	39436	5025	5
(7, *, *, 11)	4	4	0	2	(9, *, *, 17)	43870	38007	5863	6
(7, *, *, 12)	3	3	0	2	(9, *, *, 18)	38493	32492	6001	5
(7, *, *, 13)	1	1	0	1	(9, *, *, 19)	30216	24741	5475	5
(7, *, *, 14)	1	1	0	1	(9, *, *, 20)	21089	16747	4342	4
(7, *, *, *)	31	31	0	15	(9, *, *, 21)	13231	10069	3162	4
(8, *, *, 6)	1	1	0	1	(9, *, *, 22)	7181	5306	1875	3
(8, *, *, 7)	5	5	0	2	(9, *, *, 23)	3604	2468	1136	3
(8, *, *, 8)	27	27	0	3	(9, *, *, 24)	1390	946	444	2
(8, *, *, 9)	76	76	0	4	(9, *, *, 25)	567	331	236	2
(8, *, *, 10)	138	137	1	4	(9, *, *, 26)	121	76	45	1
(8, *, *, 11)	209	205	4	3	(9, *, *, 27)	50	23	27	1
(8, *, *, 12)	231	225	6	4	(9, *, *, *)	316014	274148	41866	88
(8, *, *, 13)	226	218	8	3					
(8, *, *, 14)	173	166	7	4					
(8, *, *, 15)	122	117	5	3					
(8, *, *, 16)	70	65	5	3					
(8, *, *, 17)	33	31	2	2					
(8, *, *, 18)	16	14	2	2					
(8, *, *, 19)	5	4	1	1					
(8, *, *, 20)	4	3	1	1					
(8, *, *, *)	1336	1294	42	40					

Table 6. Combinatorial 3-spheres and 4-polytopes with ≤ 8 vertices, grouped by f -vector

f -vector	3-spheres	4-polytopes	non-realizable	# of flag f -vectors
(5, 10, 10, 5)	1	1	0	1
(6, 13, 13, 6)	1	1	0	1
(6, 14, 15, 7)	1	1	0	1
(6, 14, 16, 8)	1	1	0	1
(6, 15, 18, 9)	1	1	0	1
(7, 15, 14, 6)	1	1	0	1
(7, 16, 16, 7)	2	2	0	1
(7, 17, 17, 7)	1	1	0	1
(7, 17, 18, 8)	4	4	0	2
(7, 18, 19, 8)	1	1	0	1
(7, 17, 19, 9)	1	1	0	1
(7, 18, 20, 9)	6	6	0	2
(7, 18, 21, 10)	4	4	0	2
(7, 19, 22, 10)	2	2	0	1
(7, 18, 22, 11)	1	1	0	1
(7, 19, 23, 11)	3	3	0	2
(7, 19, 24, 12)	2	2	0	1
(7, 20, 25, 12)	1	1	0	1
(7, 20, 26, 13)	1	1	0	1
(7, 21, 28, 14)	1	1	0	1
(8, 16, 14, 6)	1	1	0	1
(8, 18, 17, 7)	4	4	0	2
(8, 19, 18, 7)	1	1	0	1
(8, 19, 19, 8)	13	13	0	2
(8, 20, 20, 8)	12	12	0	2
(8, 21, 21, 8)	2	2	0	1
(8, 19, 20, 9)	1	1	0	1
(8, 20, 21, 9)	31	31	0	2
(8, 21, 22, 9)	37	37	0	3
(8, 22, 23, 9)	7	7	0	2
(8, 20, 22, 10)	7	7	0	2
(8, 21, 23, 10)	71	71	0	3
(8, 22, 24, 10)	57	56	1	3
(8, 23, 25, 10)	3	3	0	1
(8, 21, 24, 11)	26	26	0	3
(8, 22, 25, 11)	129	128	1	4
(8, 23, 26, 11)	54	51	3	2
(8, 21, 25, 12)	4	4	0	1
(8, 22, 26, 12)	75	75	0	4
(8, 23, 27, 12)	133	129	4	3
(8, 24, 28, 12)	19	17	2	1
(8, 22, 27, 13)	16	16	0	2
(8, 23, 28, 13)	113	112	1	3
(8, 24, 29, 13)	97	90	7	2
(8, 22, 28, 14)	3	3	0	1
(8, 23, 29, 14)	30	30	0	2
(8, 24, 30, 14)	105	103	2	3
(8, 25, 31, 14)	35	30	5	1
(8, 23, 30, 15)	5	5	0	1
(8, 24, 31, 15)	39	39	0	2
(8, 25, 32, 15)	78	73	5	2
(8, 24, 32, 16)	8	8	0	1
(8, 25, 33, 16)	33	32	1	2
(8, 26, 34, 16)	29	25	4	1
(8, 25, 34, 17)	8	8	0	1
(8, 26, 35, 17)	25	23	2	2
(8, 26, 36, 18)	6	6	0	1
(8, 27, 37, 18)	10	8	2	1
(8, 27, 38, 19)	5	4	1	1
(8, 28, 40, 20)	4	3	1	1

Table 7. Combinatorial 3-spheres and 4-polytopes with 9 vertices, grouped by f -vector

f -vector	3-spheres	4-polytopes	non-realizable	# of flag f -vectors	f -vector	3-spheres	4-polytopes	non-realizable	# of flag f -vectors
(9, 18, 15, 6)	1	1	0	1	(9, 26, 33, 16)	96	96	0	2
(9, 19, 17, 7)	1	1	0	1	(9, 27, 34, 16)	2038	2035	3	4
(9, 20, 18, 7)	6	6	0	2	(9, 28, 35, 16)	13869	13440	429	5
(9, 20, 19, 8)	1	1	0	1	(9, 29, 36, 16)	22973	20057	2916	4
(9, 21, 20, 8)	31	31	0	2	(9, 30, 37, 16)	5485	3808	1677	2
(9, 22, 21, 8)	37	37	0	3	(9, 26, 34, 17)	7	7	0	1
(9, 23, 22, 8)	7	7	0	2	(9, 27, 35, 17)	268	268	0	2
(9, 20, 20, 9)	1	1	0	1	(9, 28, 36, 17)	4077	4047	30	4
(9, 22, 22, 9)	129	129	0	3	(9, 29, 37, 17)	19345	18090	1255	4
(9, 23, 23, 9)	211	209	2	3	(9, 30, 38, 17)	18645	14763	3882	3
(9, 24, 24, 9)	118	116	2	3	(9, 31, 39, 17)	1528	832	696	1
(9, 25, 25, 9)	7	7	0	2	(9, 27, 36, 18)	23	23	0	1
(9, 26, 26, 9)	1	1	0	1	(9, 28, 37, 18)	596	596	0	2
(9, 22, 23, 10)	12	12	0	3	(9, 29, 38, 18)	6671	6519	152	4
(9, 23, 24, 10)	398	397	1	4	(9, 30, 39, 18)	21049	18482	2567	4
(9, 24, 25, 10)	904	897	7	4	(9, 31, 40, 18)	10154	6872	3282	2
(9, 25, 26, 10)	524	504	20	4	(9, 28, 38, 19)	45	45	0	1
(9, 26, 27, 10)	67	62	5	3	(9, 29, 39, 19)	1061	1057	4	2
(9, 23, 25, 11)	66	65	1	4	(9, 30, 40, 19)	9073	8578	495	4
(9, 24, 26, 11)	1188	1185	3	4	(9, 31, 41, 19)	17202	13559	3643	3
(9, 25, 27, 11)	2650	2593	57	4	(9, 32, 42, 19)	2835	1502	1333	1
(9, 26, 28, 11)	1344	1266	78	4	(9, 29, 40, 20)	84	84	0	1
(9, 27, 29, 11)	125	107	18	3	(9, 30, 41, 20)	1601	1574	27	2
(9, 28, 30, 11)	3	2	1	1	(9, 31, 42, 20)	9905	8793	1112	3
(9, 23, 26, 12)	3	3	0	1	(9, 32, 43, 20)	9499	6296	3203	2
(9, 24, 27, 12)	335	333	2	5	(9, 30, 42, 21)	128	128	0	1
(9, 25, 28, 12)	3275	3250	25	4	(9, 31, 43, 21)	2114	2016	98	2
(9, 26, 29, 12)	5928	5662	266	5	(9, 32, 44, 21)	8281	6536	1745	3
(9, 27, 30, 12)	2171	1943	228	4	(9, 33, 45, 21)	2708	1389	1319	1
(9, 28, 31, 12)	113	86	27	2	(9, 31, 44, 22)	175	172	3	1
(9, 24, 28, 13)	33	33	0	2	(9, 32, 45, 22)	2298	2064	234	2
(9, 25, 29, 13)	1223	1219	4	5	(9, 33, 46, 22)	4708	3070	1638	2
(9, 26, 30, 13)	7677	7536	141	6	(9, 32, 46, 23)	223	212	11	1
(9, 27, 31, 13)	9773	9023	750	5	(9, 33, 47, 23)	1976	1563	413	2
(9, 28, 32, 13)	2224	1829	395	3	(9, 34, 48, 23)	1405	693	712	1
(9, 29, 33, 13)	45	26	19	1	(9, 33, 48, 24)	231	209	22	1
(9, 25, 30, 14)	205	205	0	3	(9, 34, 49, 24)	1159	737	422	2
(9, 26, 31, 14)	3624	3608	16	6	(9, 34, 50, 25)	209	163	46	1
(9, 27, 32, 14)	14312	13744	568	5	(9, 35, 51, 25)	358	168	190	1
(9, 28, 33, 14)	11714	10268	1446	4	(9, 35, 52, 26)	121	76	45	1
(9, 29, 34, 14)	1379	996	383	2	(9, 36, 54, 27)	50	23	27	1
(9, 25, 31, 15)	15	15	0	1					
(9, 26, 32, 15)	771	771	0	4					
(9, 27, 33, 15)	7977	7878	99	5					
(9, 28, 34, 15)	20764	19241	1523	5					
(9, 29, 35, 15)	9961	7984	1977	3					
(9, 30, 36, 15)	387	216	171	1					

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